BASICS OF SAR POLARIMETRY I

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Basics of Radar Polarimetry

Abstract A comprehensive overview of the basic principles of radar polarimetry is presented. The relevant fundamental field equations are first provided. The importance of the propagation and scattering behavior in various frequency bands, the electrodynamic foundations such as Maxwell's equations, the Helmholtz vector wave equation and especially the fundamental laws of polarization will first be introduced: The fundamental terms which represent the polarization state will be introduced, defined and explained. Main points of view are the polarization Ellipse, the polarization ratio, the Stokes Parameter and the Stokes and Jones vector formalisms as well as its presentation on the Poincaré sphere and on relevant map projections. The Polarization Fork descriptor and the associated van Zyl polarimetric power density and Agrawal polarimetric phase correlation signatures will be introduced also in order to make understandable the polarization state formulations of electromagnetic waves in the frequency domain. The polarization state of electromagnetic waves under scattering conditions i.e. in the radar case will be described by matrix formalisms. Each scatterer is a polarization transformer; under normal conditions the transformation from the transmitted wave vector to the received wave vector is linear and this behavior, principally, will be described by a matrix called scattering matrix. This matrix contains all the information about the scattering process and the scatterer itself. The different relevant matrices, the respective terms like Jones Matrix, S-matrix, Müller M-matrix, Kennaugh K-matrix, etc. and its interconnections will be defined and described together with change of polarization bases transformation operators, where upon the optimal (Characteristic) polarization states are determined for the coherent and partially coherent cases, respectively. The lecture is concluded with a set of simple examples.

1. Introduction: A Review of Polarimetry

Radar Polarimetry (*Polar*: polarization, *Metry*: measure) is the science of acquiring, processing and analyzing the polarization state of an electromagnetic field. Radar polarimetry is concerned with the utilization of polarimetry in radar applications as reviewed most recently in Boerner [1] where a host of pertinent references are provided. Although polarimetry has a long history which reaches back to the 18th century, the earliest work that is related to radar dates back to the 1940s. In 1945 G.W. Sinclair introduced the concept of the scattering matrix as a descriptor of the radar cross section of a coherent scatterer [2], [3]. In the late 1940s and the early 1950s major pioneering work was carried out by E.M. Kennaugh [4, 5]. He formulated a backscatter theory based on the eigenpolarizations of the scattering matrix introducing the concept of optimal polarizations by implementing the concurrent work of G.A. Deschamps, H. Mueller, and C. Jones. Work continued after Kennaugh, but only a few notable contributions, as those of G.A. Deschamps 1951 [6], C.D. Graves 1956 [7], and J.R. Copeland 1960 [8], were made until Huynen's studies in 1970s. The beginning of a new age was the treatment presented by J.R. Huynen in his doctoral thesis of 1970 [9], where he exploited Kennaugh's optimal polarization concept [5] and formulated his approach to target radar phenomenology. With this thesis, a renewed interest for radar polarimetry was raised. However, the full potential of radar polarimetry was never fully realized until the early 1980s, due in no small parts to the advanced radar device technology [10, 11]. Technological problems led to a series of negative conclusions in the 1960s and 1970s about the practical use of radar systems with polarimetric capability [12]. Among the major contributions of the 1970s and 1980s are those of W-M Boerner [13, 14, 15] who pointed out the importance of polarization first in addressing vector electromagnetic inverse scattering [13]. He initiated a critical analysis of Kennaugh's and Huynen's work and extended Kennaugh's optimal polarization theory [16]. He has been influential in causing the radar community to recognize the need of polarimetry in remote sensing applications. A detailed overview on the history of polarimetry can be found in [13, 14, 15], while a historical review of polarimetric radar technology is also given in [13, 17, 18].

Boerner, W.-M. (2007) Basics of SAR Polarimetry I. In *Radar Polarimetry and Interferometry* (pp. 3-1 – 3-40). Educational Notes RTO-EN-SET-081bis, Paper 3. Neuilly-sur-Seine, France: RTO. Available from: http://www.rto.nato.int/abstracts.asp.

Polarimetry deals with the full vector nature of polarized (vector) electromagnetic waves throughout the frequency spectrum from Ultra-Low-Frequencies (ULF) to above the Far-Ultra-Violet (FUV) [19, 20]. Whenever there are abrupt or gradual changes in the index of refraction (or permittivity, magnetic permeability, and conductivity), the polarization state of a narrow band (single-frequency) wave is transformed, and the electromagnetic "vector wave" is re-polarized. When the wave passes through a medium of changing index of refraction, or when it strikes an object such as a radar target and/or a scattering surface and it is reflected; then, characteristic information about the reflectivity, shape and orientation of the reflecting body can be obtained by implementing '*polarization control*' [10, 11]. The complex direction of the electric field vector, in general describing an ellipse, in a plane transverse to propagation, plays an essential role in the interaction of electromagnetic '*vector waves*' with material bodies, and the propagation medium [21, 22, 13, 14, 16]. Whereas, this polarization transformation behavior, expressed in terms of the "polarization ellipse" is named "*Ellipsometry*" in Optical Sensing and Imaging [21, 23], it is denoted as *"Polarimetry*" in Radar, Lidar/Ladar and SAR Sensing and Imaging [12, 14, 15, 19] - using the ancient Greek meaning of "*measuring orientation and object shape*". Thus, *ellipsometry* and *polarimetry* are concerned with the control of the coherent polarization properties of the optical and radio waves, respectively [21, 19]. With the advent of optical and radar polarization phase control devices, *ellipsometry* advanced rapidly during the Forties (Mueller and Land [24, 21]) with the associated development of mathematical *ellipsometry*, i.e., the introduction of '*the 2 x 2 coherent Jones forward scattering (propagation) and the associated 4 x 4 average power density Mueller (Stokes) propagation matrices*' [21]; and *polarimetry* developed independently in the late Forties with the introduction of dual polarized antenna technology (Sinclair, Kennaugh, et al. [2, 3, 4, 5]), and the subsequent formulation of *'the 2 x 2 coherent Sinclair radar back-scattering matrix and the associated 4 x 4 Kennaugh radar back-scattering power density matrix'*, as summarized in detail in Boerner et al. [19, 25]. Since then, *ellipsometry* and *polarimetry* have enjoyed steep advances; and, a mathematically coherent polarization matrix formalism is in the process of being introduced for which the lexicographic covariance matrix presentations [26, 27] of signal estimation theory play an equally important role in *ellipsometry* as well as *polarimetry* [19]. Based on Kennaugh's original pioneering work on discovering the properties of the "Spinorial Polarization Fork" concept [4, 5], Huynen [9] developed a *"Phenomenological Approach to Radar Polarimetry",* which had a subtle impact on the steady advancement of *polarimetry* [13, 14, 15] as well as *ellipsometry* by developing the "*orthogonal (group theoretic) target scattering matrix decomposition*" [28, 29, 30] and by extending the characteristic optimal polarization state concept of Kennaugh [31, 4, 5], which lead to the renaming of the spinorial polarization fork concept to the so called '*Huynen Polarization Fork*' in '*Radar Polarimetry*' [31]. Here, we emphasize that for treating the general bistatic (asymmetric) scattering matrix case, a more general formulation of fundamental *Ellipsometry* and *Polarimetry* in terms of a spinorial group-theoretic approach is strictly required, which was first explored by Kennaugh but not further pursued by him due to the lack of pertinent mathematical formulations [32, 33].

In *ellipsometry*, the Jones and Mueller matrix decompositions rely on a product decomposition of relevant optical measurement/transformation quantities such as diattenuation, retardence, depolarization, birefringence, etc., [34, 35, 23, 28, 29] measured in a '*chain matrix arrangement, i.e., multiplicatively placing one optical decomposition device after the other*'. In *polarimetry*, the Sinclair, the Kennaugh, as well as the covariance matrix decompositions [29] are based on a group-theoretic series expansion in terms of the principal orthogonal radar calibration targets such as the sphere or flat plate, the linear dipole and/or circular helical scatterers, the dihedral and trihedral corner reflectors, and so on - - observed in a linearly superimposed aggregate measurement arrangement [36, 37]; leading to various canonical target feature mappings [38] and sorting as well as scatter-characteristic decomposition theories [39, 27, 40]. In addition, polarization-dependent speckle and noise reduction play an important role in both *ellipsometry* and *polarimetry, which in radar polarimetry were first pursued with rigor by J-S. Lee* [41, 42, 43, 44]. The implementation of all of these novel methods will fail unless one is given fully calibrated scattering matrix information, which applies to each element of the Jones and Sinclair matrices.

It is here noted that it has become common usage to replace "ellipsometry" by "optical polarimetry" and expand "polarimetry" to "radar polarimetry" in order to avoid confusion [45, 18], a nomenclature adopted in the remainder of this paper.

Very remarkable improvements beyond classical "non-polarimetric" radar target detection, recognition and discrimination, and identification were made especially with the introduction of the covariance matrix optimization procedures of Tragl [46], Novak et al. [47 - 51], Lüneburg [52 - 55], Cloude [56], and of Cloude and Pottier [27]. Special attention must be placed on the *'Cloude-Pottier Polarimetric Entropy H, Anisotropy A, Feature-Angle ($\overline{\alpha}$) parametric decomposition'* [57] because it allows for unsupervised target feature interpretation [57, 58]. Using the various fully polarimetric (scattering matrix) target feature syntheses [59], polarization contrast optimization, [60, 61] and polarimetric entropy/anisotropy classifiers, very considerable progress was made in interpreting and analyzing POL-SAR image features [62, 57, 63, 64, 65, 66]. This includes the reconstruction of *'Digital Elevation Maps (DEMs)'* directly from *'POL-SAR Covariance-Matrix Image Data Takes'* [67 - 69] next to the familiar method of DEM reconstruction from IN-SAR Image data takes [70, 71, 72]. In all of these techniques well calibrated scattering matrix data takes are becoming an essential pre-requisite without which little can be achieved [18, 19, 45, 73]. In most cases the *'multi-look-compressed SAR Image data take MLC- formatting'* suffices also for completely polarized SAR image algorithm implementation [74]. However, in the sub-aperture polarimetric studies, in '*Polarimetric SAR Image Data Take Calibration*', and in '*POL-IN-SAR Imaging*', the '*SLC (Single Look Complex) SAR Image Data Take Formatting'* becomes an absolute must [19, 1]. Of course, for SLC-formatted Image data, in particular, various speckle-filtering methods must be applied always. Implementation of the '*Lee Filter'* – explored first by Jong-Sen Lee - for speckle reduction in polarimetric SAR image reconstruction, and of the *' Polarimetric Lee-Wishart distribution'* for improving image feature characterization have further contributed toward enhancing the interpretation and display of high quality SAR Imagery [41 – 44, 75].

2. The Electromagnetic Vector Wave and Polarization Descriptors

The fundamental relations of radar polarimetry are obtained directly from Maxwell's equations [86, 34], where for the source-free isotropic, homogeneous, free space propagation space, and assuming IEEE standard [102] time-dependence $\exp(+j\omega t)$, the electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields satisfy with μ being the free space permeability and ε the free space permittivity

$$\nabla \mathrm{x}\mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{H}(\mathbf{r}), \qquad \nabla \mathrm{x}\mathbf{H}(\mathbf{r}) = j\omega\varepsilon\, \mathbf{E}(\mathbf{r}) \tag{2.1}$$

which for the time-invariant case, result in

$$(\nabla + k^2)\mathbf{E} = 0, \qquad \mathbf{E}(\mathbf{r}) = E_0 \frac{\exp(-jkr)}{r}, \qquad \mathbf{H}(\mathbf{r}) = H_0 \frac{\exp(-jkr)}{r} \tag{2.2}$$

for an outgoing spherical wave with propagation constant $k = \omega\ (\varepsilon\ \mu)^{1/2}$ and $c = (\varepsilon\ \mu)^{-1/2}$ being the free space velocity of electromagnetic waves

No further details are presented here, and we refer to Stratton [86], Born and Wolf [34] and Mott [76] for full presentations.

2.1 Polarization Vector and Complex Polarization Ratio

With the use of the standard spherical coordinate system $(r,\theta,\phi\ ;\hat{\mathrm{u}}_r,\hat{\mathrm{u}}_\theta,\hat{\mathrm{u}}_\phi)$ with r,θ,ϕ denoting the radial, polar, azimuthal coordinates, and $\hat{\mathrm{u}}_r,\hat{\mathrm{u}}_\theta,\hat{\mathrm{u}}_\phi$ the corresponding unit vectors, respectively; the outward travelling wave is expressed as

$$\mathbf{E} = \hat{\mathrm{u}}_\theta\ E_\theta + \hat{\mathrm{u}}_\phi E_\phi \quad \mathbf{H} = \hat{\mathrm{u}}_\theta\ H_\theta + \hat{\mathrm{u}}_\phi\ H_\phi\ ,\ \mathbf{P} = \frac{\hat{\mathrm{u}}_r}{2}\left|\mathbf{E}\times\mathbf{H}^*\right| = \frac{\hat{\mathrm{u}}_r\ |\mathrm{E}|^2}{2Z_0}\ ,\ Z_0 = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} = 120\pi\ [\Omega] \tag{2.3}$$

with $\mathbf{P}$ denoting the Poynting power density vector, and Z_0 being the intrinsic impedance of the medium (here vacuum). Far from the antenna in the far field region [86, 76], the radial waves of (2.2) take on plane wave characteristics, and assuming the wave to travel in positive z-direction of a right-handed Cartesian coordinate system (x, y, z), the electric field $\mathbf{E}$, denoting the polarization vector, may be rewritten as

$$\mathbf{E} = \hat{u}_x E_x + \hat{u}_y E_y = |E_x| \exp(j\phi_x)\{\hat{u}_x + \hat{u}_y \left|\frac{E_y}{E_x}\right| \exp(j\phi)\} \tag{2.4}$$

with $|E_x|, |E_y|$ being the amplitudes, ϕ_x, ϕ_y the phases, $\phi = \phi_y - \phi_x$ the relative phase; $|E_x / E_y| = \tan\alpha$ with ϕ_x, ϕ_y, α and ϕ defining the Deschamps parameters [6, 103]. Using these definitions, the '*normalized complex polarization vector* $\mathbf{p}$' and the '*complex polarization ratio* ρ' can be defined as

$$\mathbf{p} = \frac{\mathbf{E}}{|\mathbf{E}|} = \frac{\hat{u}_x E_x + \hat{u}_y E_y}{|\mathbf{E}|} = \frac{E_x}{|\mathbf{E}|}\left(\hat{u}_x + \rho\, \hat{u}_y\right) \tag{2.5}$$

with $|\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E}^* = E_x^{\;2} + E_y^{\;2}$ and $|\mathbf{E}| = E$ defines the wave amplitude, and ρ is given by

$$\rho = \frac{E_y}{E_x} = \left|\frac{E_y}{E_x}\right| \exp(j\phi), \qquad \phi = \phi_y - \phi_x \tag{2.6}$$

2.2 The Polarization Ellipse and its Parameters

The tip of the real time-varying vector $\mathbf{E}$, or $\mathbf{p}$, traces an ellipse for general phase difference ϕ, where we distinguish between right-handed (clockwise) and left-handed (counter-clockwise) when viewed by the observer in direction of the travelling wave [76, 19], as shown in Fig. 2.1 for the commonly used horizontal H (by replacing x) and vertical V (by replacing y) polarization states.

There exist unique relations between the alternate representations, as defined in Fig. 2.1 and Fig. 2.2 with the definition of the orientation ψ and ellipticity χ angles expressed, respectively, as

$$\alpha = |\rho| = \frac{|E_y|}{|E_x|},\ 0 \le \alpha \le \pi/2 \quad \text{and} \quad \tan 2\psi = \tan(2\alpha)\cos\phi \qquad -\pi/2 \le \psi \le +\pi/2 \tag{2.7}$$

$$\tan\chi = \pm \textit{minor axis/major axis}, \qquad \sin 2\chi = \sin 2\alpha \sin\phi\ , \qquad -\pi/4 \le \chi \le \pi/4 \tag{2.8}$$

where the $+$ and $-$ signs are for left- and right-handed polarizations respectively.

For a pair of orthogonal polarizations $\mathbf{p_1}$ and $\mathbf{p_2} = \mathbf{p_{1_\perp}}$

$$\mathbf{p_1} \cdot \mathbf{p_2}^* = 0 \qquad \rho_2 = \rho_{1\perp} = -1/\rho_1^* \ , \qquad \psi_1 = \psi_2 + \frac{\pi}{2} \quad \chi_1 = -\chi_2 \tag{2.9}$$

In addition, the following useful transformation relations exist:

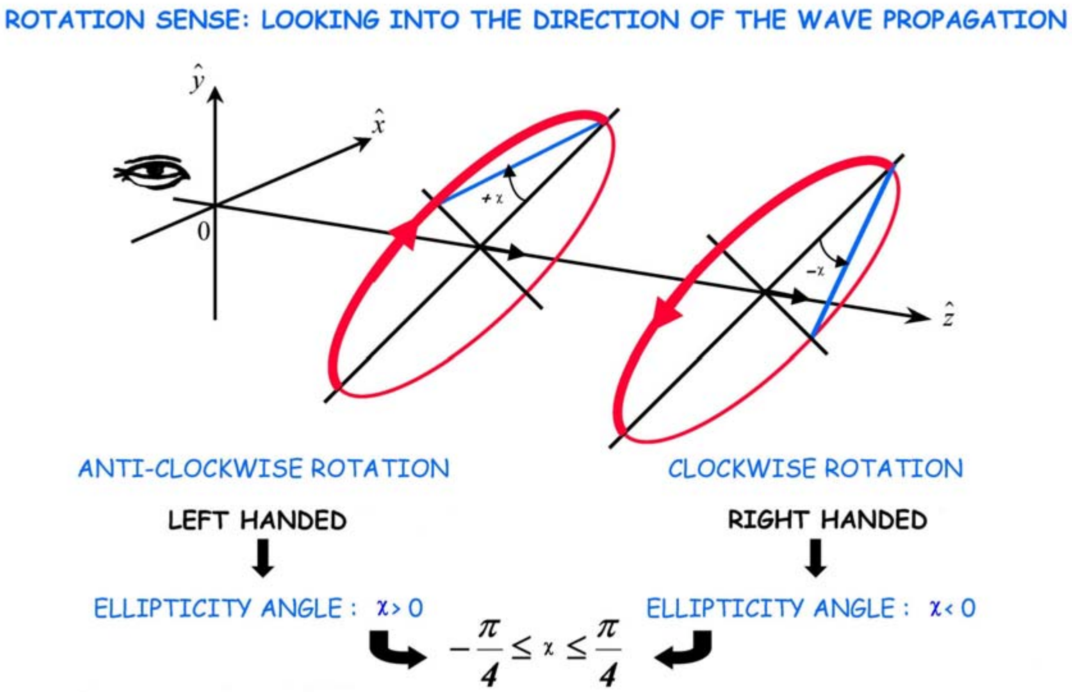
$$\rho = \frac{\cos 2\chi \sin 2\psi + j \sin 2\chi}{1 + \cos 2\chi \cos 2\psi} = \tan\alpha \exp(j\phi) \tag{2.10}$$

where (α,ϕ) and (ψ,χ) are related by the following equations:

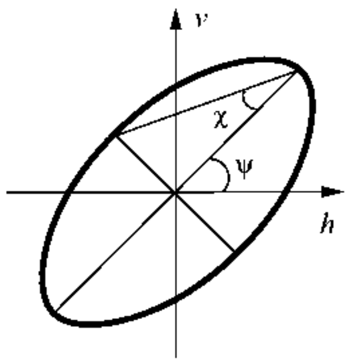
$$\cos 2\alpha = \cos 2\psi \cos 2\chi, \quad \tan\phi = \tan 2\chi / \sin 2\psi \tag{2.11}$$

and inversely

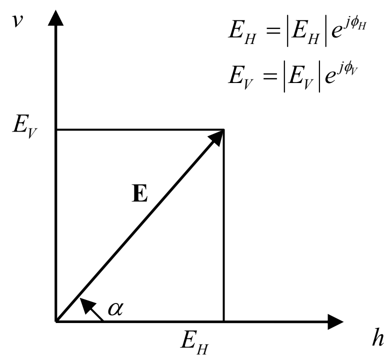
$$\psi = \frac{1}{2}\arctan\left(\frac{2\,\mathrm{Re}\{\rho\}}{1-\rho\rho^*}\right) + \pi \quad \textbf{... mod}(\pi) \qquad \chi = \frac{1}{2}\arcsin\left(\frac{2\,\mathrm{Im}\{\rho\}}{1-\rho\rho^*}\right) \tag{2.12}$$



(a) Rotation Sense (Courtesy of Prof. E. Pottier)



(b) Orientation ψ and Ellipticity χ Angles.



(c) Electric Field Vector.

Fig. 2.1 Polarization Ellipse.

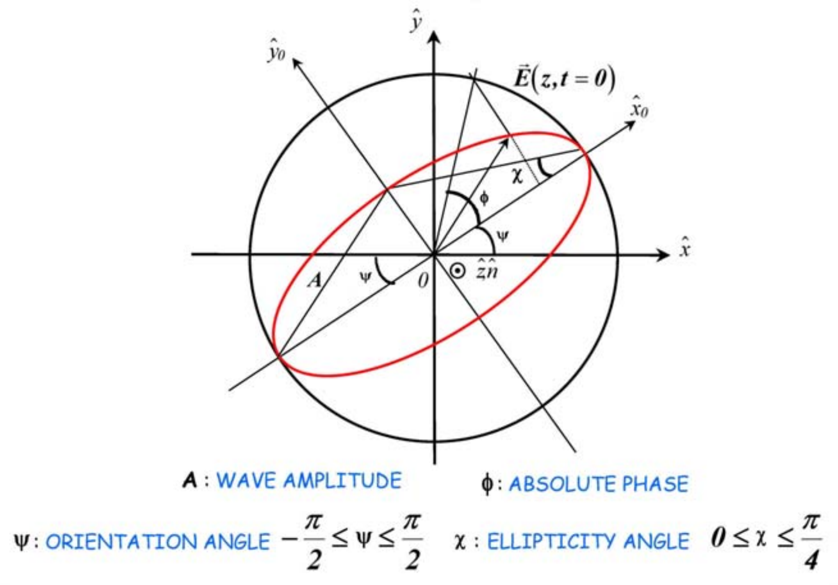


Fig. 2.2 Polarization Ellipse Relations (Courtesy of Prof. E. Pottier)

Another useful formulation of the polarization vector **p** was introduced by Huynen in terms of the parametric formulation [9, 104], derived from group-theoretic considerations based on the Pauli SU(2) matrix set $\psi_P\left\{[\sigma_i]\ ,\ i=0,1,2,3\right\}$ as further pursued by Pottier [105], where according to (2.10) and (2.11), for $\psi = 0$, and then rotating this ellipse by ψ.

$$\mathbf{p}(|\mathbf{E}|,\phi,\psi,\chi) = |\mathbf{E}|\exp(j\phi)\begin{bmatrix}\cos\psi & -\sin\psi \\ \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\cos\chi \\ -j\sin\chi\end{bmatrix} \tag{2.13}$$

which will be utilized later on; and $\psi_P\left\{[\sigma_i]\ ,\ i=0,1,2,3\right\}$ is defined in terms of the classical unitary Pauli matrices $[\sigma_i]$ as

$$[\sigma_0]=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix},\quad [\sigma_1]=\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix},\quad [\sigma_2]=\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix},\quad [\sigma_3]=\begin{bmatrix}0 & -j \\ j & 0\end{bmatrix} \tag{2.14}$$

where the $[\sigma_i]$ matrices satisfy the unitarity condition as well as commutation properties given by

$$[\sigma_i]^{-1} = [\sigma_i]^{T*},\quad \left|Det\left\{[\sigma_i]\right\}\right| = 1,\quad [\sigma_i][\sigma_j] = -[\sigma_j][\sigma_i],\quad [\sigma_i][\sigma_i] = [\sigma_0] \tag{2.15}$$

satisfying the ordinary matrix product relations.

2.3 <u>The Jones Vector and Changes of Polarization Bases</u>

If instead of the basis {x y} or {H V}, we introduce an alternative presentation {m n} as a linear combination of two arbitrary orthonormal polarization states $\mathbf{E}_m$ and $\mathbf{E}_n$ for which

$$\mathbf{E} = \hat{\mathbf{u}}_m E_m + \hat{\mathbf{u}}_n E_n \tag{2.16}$$

and the standard basis vectors are in general, orthonormal, i.e.

$$\hat{u}_m \cdot \hat{u}_n^\dagger = 0, \quad \hat{u}_m \cdot \hat{u}_m^\dagger = \hat{u}_n \cdot \hat{u}_n^\dagger = 1 \tag{2.17}$$

with $\dagger$ denoting the hermitian adjoint operator [21, 52, 53]; and the Jones vector $\mathbf{E}_{mn}$ may be defined as

$$\mathbf{E}_{mn} = \begin{bmatrix} E_m \\ E_n \end{bmatrix} = \begin{bmatrix} |E_m| \exp j\phi_m \\ |E_n| \exp j\phi_n \end{bmatrix} = E_m \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \frac{|\mathbf{E}| \exp(j\phi_m)}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} 1 \\ \rho \end{bmatrix} = |\mathbf{E}| \exp(j\phi_m) \begin{bmatrix} \cos\alpha \\ \sin\alpha \exp(j\phi) \end{bmatrix} \tag{2.18}$$

with $\tan\alpha = | E_n / E_m |$ and $\phi = \phi_n - \phi_m$. This states that the Jones vector possesses, in general, four degrees of freedom. The Jones vector descriptions for characteristic polarization states are provided in Fig. 2.3.

$$\mathbf{E}_{mn} = \mathbf{E}(m,n) = \begin{bmatrix} E_m \\ E_n \end{bmatrix} \qquad \mathbf{E}_{ij} = \mathbf{E}(i,j) = \begin{bmatrix} E_i \\ E_j \end{bmatrix} \quad \text{and} \quad \mathbf{E}_{AB} = \mathbf{E}(A,B) = \begin{bmatrix} E_A \\ E_B \end{bmatrix} \tag{2.20}$$

The unique transformation from the $\{\hat{\mathbf{u}}_m \ \hat{\mathbf{u}}_n\}$ to the arbitrary $\{\hat{\mathbf{u}}_i \ \hat{\mathbf{u}}_j\}$ or $\{\hat{\mathbf{u}}_A \ \hat{\mathbf{u}}_B\}$ bases is sought which is a linear transformation in the two-dimensional complex space so that

$$\mathbf{E}_{ij} = [U_2]\mathbf{E}_{mn} \quad or \quad \mathbf{E}(i,j) = [U_2]\mathbf{E}(m,n) \quad \text{with} \quad [U_2]\,[U_2]^\dagger = [I_2] \tag{2.21}$$

satisfying wave energy conservation with $[I_2]$ being the 2x2 identity matrix, and we may choose, as shown in [81],

$$\hat{\mathbf{u}}_i = \frac{\exp(j\phi_i)}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} 1 \\ \rho \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_{i_\perp} = \frac{\exp(j\phi_i)}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} 1 \\ -\rho^{*^{-1}} \end{bmatrix} = \frac{\exp(j\phi_i)}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} -\rho^* \\ 1 \end{bmatrix} \tag{2.22}$$

with $\phi_j' = \phi_j + \phi + \pi$ so that

$$[U_2] = \frac{1}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} \exp(j\phi_i) & -\rho^* \exp(j\phi_j) \\ \rho \exp(j\phi_i) & \exp(j\phi_j) \end{bmatrix} \tag{2.23}$$

yielding $Det\{[U_2]\} = \exp\{j(\phi_i + \phi_j')\}$ with $\phi_i + \phi_j{}' = 0$

Since any monochromatic plane wave can be expressed as a linear combination of two orthonormal linear polarization states, defining the reference polarization basis, there exist an infinite number of such bases {i j} or {A B} for which

$$\mathbf{E} = \hat{u}_m E_m + \hat{u}_n E_n = \hat{u}_i E_i + \hat{u}_j E_j = \hat{u}_A E_A + \hat{u}_B E_B \tag{2.19}$$

with corresponding Jones vectors presented in two alternate, most commonly used notations

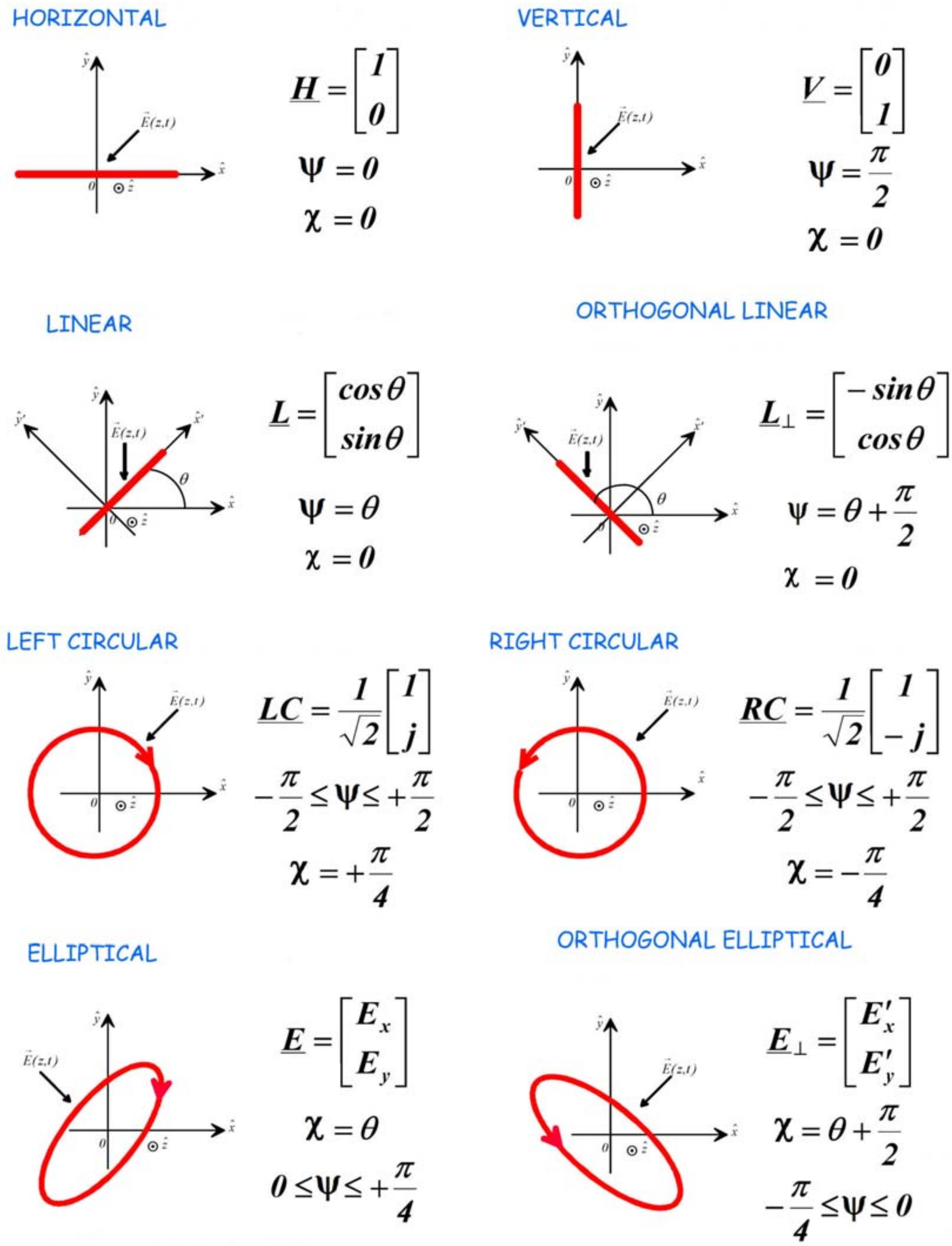


Fig. 2.3 Jones Vector Descriptions for Characteristic Polarization States with direction of propagation out of the page (Courtesy of Prof. E. Pottier)

Since $[U_2]$ is a special unitary 2x2 complex matrix with unit determinant, implying that (i) the amplitude of the wave remains independent of the change of the polarization basis, and that (ii) the phase of the (absolute) wave may be consistently defined as the polarization basis is changed, we finally obtain,

$$[U_2] = \frac{1}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \exp(j\phi_i) & 0 \\ 0 & \exp(j\phi_i) \end{bmatrix} \tag{2.24}$$

possessing three degrees of freedom similar to the normalized Jones vector formulation, but in most cases the phase reference is taken as $\phi_i = 0$ which may not be so in polarimetric interferometry [96]. For further details on the group-theoretic representations of the proper transformation relations see the formulations derived by Pottier in [106].

2.4 Complex Polarization Ratio in Different Polarization Bases

Any wave can be resolved into two orthogonal components (linearly, circularly, or elliptically polarized) in the plane transverse to the direction of propagation. For an arbitrary polarization basis {A B} with unit vectors $\hat{a}$ and $\hat{b}$, one may define the polarization state

$$\mathbf{E}(AB) = E_A\,\hat{a} + E_B\,\hat{b} \tag{2.25}$$

where the two components E_A and E_B are complex numbers. The polarization ratio ρ_{AB} in an arbitrary basis {A B} is also a complex number, and it may be defined as

$$\rho_{AB} = \frac{E_B}{E_A} = \frac{|E_B|}{|E_A|}\exp\{j(\phi_B - \phi_A)\} = |\rho_{AB}|\exp\{j\phi_{AB}\} \tag{2.26}$$

where $|\rho_{AB}|$ is the ratio of magnitude of two orthogonal components of the field $|E_A|$ and $|E_B|$ and ϕ_{AB} is the phase difference between E_A and E_B. The complex polarization ratio ρ_{AB} depends on the polarization basis {A B} and can be used to specify the polarization of an electromagnetic wave

$$\mathbf{E}(AB) = \begin{bmatrix} E_A \\ E_A \end{bmatrix} = |E_A|\exp\{j\phi_A\}\begin{bmatrix} 1 \\ \rho_{AB} \end{bmatrix} = |E_A|\exp\{j\phi_A\}\frac{\sqrt{1+\frac{E_B E_B^*}{E_A E_A^*}}}{\sqrt{1+\frac{E_B E_B^*}{E_A E_A^*}}}\begin{bmatrix} 1 \\ \rho_{AB} \end{bmatrix}$$
$$= |\mathbf{E}|\exp\{j\phi_A\}\frac{1}{\sqrt{1+\rho_{AB}\rho_{AB}^*}}\begin{bmatrix} 1 \\ \rho_{AB} \end{bmatrix} \tag{2.27}$$

where $|\mathbf{E}| = \sqrt{E_A E_A^* + E_B E_B^*}$ is the amplitude of the wave $\mathbf{E}(AB)$. If we choose $|\mathbf{E}| = 1$ and disregard the absolute phase ϕ_A, the above representation becomes

$$\mathbf{E}(AB) = \frac{1}{\sqrt{1+\rho_{AB}\rho_{AB}^*}}\begin{bmatrix} 1 \\ \rho_{AB} \end{bmatrix} \tag{2.28}$$

This representation of the polarization state using the polarization ratio ρ_{AB} is very useful. For example, if we want to represent a left-handed circular (LHC) polarization state and a right-handed circular (RHC) polarization state in a linear basis {H V} using the polarization ratio. For a left-handed circular (LHC) polarization, $|E_H| = |E_V|$, $\phi_{HV} = \phi_V - \phi_H = \pi/2$, and according to (2.26), the polarization ratio ρ_{HV} is j. Using (2.28) with $\rho_{HV} = j$, we obtain for the left-handed circular (LHC) polarization

$$\mathbf{E}(HV) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix} \tag{2.29}$$

Similarly, the polarization ratio ρ_{HV} of a right-handed circular (RHC) polarization state in a linear basis {H V} is $-j$ because the relative phase $\phi_{HV} = -\pi/2$, and its representation is

$$\mathbf{E}(HV) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix} \tag{2.30}$$

The complex polarization ratio ρ is important in radar polarimetry. However, the value of the polarization ratio ρ defined in a certain polarization basis is different from that defined in the other polarization basis even if the physical polarization state is the same.

2.4.1 Complex Polarization Ratio in the Linear Basis {H V}

In the linear {H V} basis with unit vectors $\hat{\mathrm{h}}$ and $\hat{\mathrm{v}}$, a polarization state may be expressed as:

$$\mathbf{E}(HV) = E_H\,\hat{\mathrm{h}} + E_V\,\hat{\mathrm{v}} \tag{2.31}$$

The polarization ratio ρ_{HV}, according to (2.6), can be described as:

$$\rho_{HV} = \frac{E_V}{E_H} = \left|\frac{E_V}{E_H}\right| \exp\left(j\phi_{HV}\right) = \tan\alpha_{HV}\exp\left(j\phi_{HV}\right), \qquad \phi_{HV} = \phi_V - \phi_H \tag{2.32}$$

where the angle α_{HV} is defined in Fig. 2.1c, only in the {H V} basis and

$$\begin{aligned} |E_H| &= \sqrt{E_H^2 + E_V^2}\,\cos\alpha_{HV} \\ |E_V| &= \sqrt{E_H^2 + E_V^2}\,\sin\alpha_{HV} \end{aligned} \tag{2.33}$$

Also, for a single monochromatic, uniform TEM (transverse electromagnetic) traveling plane wave in the positive z direction, the real instantaneous electric field is written as

$$\boldsymbol{\varepsilon}(z,t) = \begin{bmatrix} \varepsilon_x(z,t) \\ \varepsilon_y(z,t) \\ \varepsilon_z(z,t) \end{bmatrix} = \begin{bmatrix} |E_x|\cos(\omega t - kz + \phi_x) \\ |E_y|\cos(\omega t - kz + \phi_y) \\ 0 \end{bmatrix} \tag{2.34}$$

In a cartesian coordinate system, the $+x$-axis is commonly chosen as the horizontal basis (H) and the $+y$-axis as the vertical basis (V) Substituting (2.33) into (2.34), we find

$$\begin{aligned} \boldsymbol{\varepsilon}(z,t) &= \begin{bmatrix} \sqrt{E_H^2 + E_V^2}\,\cos\alpha_{HV}\,\cos(\omega t - kz + \phi_H) \\ \sqrt{E_H^2 + E_V^2}\,\sin\alpha_{HV}\,\cos(\omega t - kz + \phi_V) \end{bmatrix} = \\ &= \sqrt{E_H^2 + E_V^2}\,\exp\left\{\begin{bmatrix} \cos\alpha_{HV} \\ \sin\alpha_{HV}\,\exp(j\phi) \end{bmatrix} \exp\{j(\omega t - kz + \phi_H)\}\right\} \end{aligned} \tag{2.35}$$

where $\phi = \phi_V - \phi_H$ is the relative phase. The expression in the square bracket is a spinor [32] which is independent of the time-space dependence of the traveling wave. The spinor parameters (α, ϕ) are easy to be located on the Poincaré sphere and can be used to represent the polarization state of a plane wave. In Fig. 2.4c, the polarization state, described by the point P_E on the Poincaré sphere, can be expressed in terms of these two angles, where $2\alpha_{HV}$ is the angle subtended by the great circle drawn from the point P_E on the equator measured from H toward V; and ϕ_{HV} is the angle between the great circle and the equator.

From equations, (2.7) and (2.8) for the {H V} basis we have

$$\begin{aligned} \sin 2\chi &= \sin 2\alpha_{HV} \sin\phi_{HV} \\ \tan 2\psi &= \tan(2\alpha_{HV})\cos\phi_{HV} \end{aligned} \tag{2.36}$$

which describes the ellipticity angle χ and the tilt or orientation angle ψ in terms of the variables α_{HV} and ϕ_{HV}. Also, from (2.11) for the {H V} basis an inverse pair that describes the α_{HV} and ϕ_{HV} in terms of χ and ψ is given in (2.37)

$$\begin{aligned} \cos 2\alpha_{HV} &= \cos 2\psi \cos 2\chi \\ \tan\phi_{HV} &= \frac{\tan 2\chi}{\sin 2\psi} \end{aligned} \tag{2.37}$$

It is convenient to describe the polarization state by either of the two set of angles (α_{HV},ϕ_{HV}) or (χ,ψ) which describe a point on the Poincaré sphere. The complex polarization ratio ρ_{HV} can be used to specify the polarization of an electromagnetic wave expressed in the {H V} basis. Some common polarization states expressed in terms of (χ,ψ), ρ, and the normalized Jones vector $\mathbf{E}$ are listed in Table 2.1 at the end of this section.

2.4.2 Complex Polarization Ratio in the Circular Basis {L R}

In the circular basis {L R}, we have two unit vectors $\hat{\mathrm{L}}$ (left-handed circular) and $\hat{\mathrm{R}}$ (right- handed circular). Any polarization of a plane wave can be expressed by

$$\mathbf{E}(LR) = E_L\,\hat{\mathrm{L}} + E_R\,\hat{\mathrm{R}} \tag{2.38}$$

A unit amplitude left-handed circular polarization has only the L component in the circular basis {L R}. It can be expressed by

$$\mathbf{E}(LR) = 1*\hat{\mathrm{L}} + 0*\hat{\mathrm{R}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.39}$$

The above representation of a unit (LHC) polarization in the circular basis {L R} is different from that in the linear basis {H V} of (2.29). Similarly, a unit amplitude right-handed circular polarization has only the R component in the circular basis {L R}

$$\mathbf{E}(LR) = 0*\hat{\mathrm{L}} + 1*\hat{\mathrm{R}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.40}$$

which is different from that in the linear {H V} basis.

The polarization ratio ρ_{LR}, according to (2.26) is

$$\rho_{LR} = \frac{E_R}{E_L} = \frac{|E_R|}{|E_L|}\exp\{j(\phi_R - \phi_L)\} = |\rho_{LR}|\exp\{j\phi_{LR}\} = \tan\alpha_{LR}\exp\{j\phi_{LR}\} \qquad (2.41)$$

where $|\rho_{LR}|$ is the ratio of magnitudes of the two orthogonal components $|E_L|$ and $|E_R|$, and ϕ_{LR} the phase difference. The angles α_{LR} and ϕ_{LR} are also easy to be found on the Poincaré sphere (see Fig. 2.6) like the angles α_{HV} and ϕ_{HV}. Some common polarization states in terms of ρ_{LR}, are listed in Table 2.1.

2.4.3 Complex Polarization Ratio in the Linear Basis {45° 135°}

In the linear {45° 135°} basis with unit vectors $\hat{45^\circ}$ and $\hat{135^\circ}$, a polarization state may be expressed as

$$\mathbf{E}(45^\circ 135^\circ) = E_{45^\circ}\,\hat{45^\circ} + E_{135^\circ}\,\hat{135^\circ} \qquad (2.42)$$

where E_{45° and E_{135° are the 45° component and the 135° component, respectively. The polarization ratio according to (2.26) is

$$\rho_{45^\circ 135^\circ} = \frac{E_{135^\circ}}{E_{45^\circ}} = \frac{|E_{135^\circ}|}{|E_{45^\circ}|}\exp\{j(\phi_{135^\circ} - \phi_{45^\circ})\} = |\rho_{45^\circ 135^\circ}|\exp\{j\phi_{45^\circ 135^\circ}\} = \tan\alpha_{45^\circ 135^\circ}\exp\{j\phi_{45^\circ 135^\circ}\} \qquad (2.43)$$

where $|\rho_{45^\circ 135^\circ}|$ is the ratio of magnitudes of the two orthogonal components $|E_{135^\circ}|$ and $|E_{45^\circ}|$, and $\phi_{45^\circ 135^\circ}$ the phase difference. The angles $\alpha_{45^\circ 135^\circ}$ and $\phi_{45^\circ 135^\circ}$ are also easy to be found on the Poincaré sphere (see Fig. 2.6)

TABLE 2.1

POLARIZATION STATES IN TERMS OF (χ, ψ), POLARIZATION RATIO ρ AND NORMALIZED JONES VECTOR $\mathbf{E}$

POLARIZATION	χ	ψ	{H V} basis		{45° 135°} basis		{L R} basis	
			ρ_{HV}	$\mathbf{E}$	$\rho_{45^\circ 135^\circ}$	$\mathbf{E}$	ρ_{LR}	$\mathbf{E}$
Linear Horizontal	0	0	0	$\begin{bmatrix}1\\0\end{bmatrix}$	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
Linear Vertical	0	$\frac{\pi}{2}$	∞	$\begin{bmatrix}0\\1\end{bmatrix}$	1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix}-j\\j\end{bmatrix}$
45° Linear	0	$\frac{\pi}{4}$	1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	0	$\begin{bmatrix}1\\0\end{bmatrix}$	j	$\frac{1}{2}\begin{bmatrix}1 & -j\\1 & j\end{bmatrix}$
135° Linear	0	$-\frac{\pi}{4}$	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$	∞	$\begin{bmatrix}0\\1\end{bmatrix}$	$-j$	$\frac{1}{2}\begin{bmatrix}-1 & -j\\-1 & j\end{bmatrix}$
Left-handed Circular	$\frac{\pi}{4}$		j	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	j	$\frac{1}{2}\begin{bmatrix}1 & j\\-1 & j\end{bmatrix}$	0	$\begin{bmatrix}1\\0\end{bmatrix}$
Right-handed Circular	$-\frac{\pi}{4}$		$-j$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	$-j$	$\frac{1}{2}\begin{bmatrix}1 & -j\\-1 & -j\end{bmatrix}$	∞	$\begin{bmatrix}0\\1\end{bmatrix}$

2.5 The Stokes Parameters

So far, we have seen completely polarized waves for which $|E_A|, |E_B|$, and ϕ_{AB} are constants or at least slowly varying functions of time. If we need to deal with partial polarization, it is convenient to use the Stokes parameters q_0, q_1, q_2 *and* q_3 introduced by Stokes in 1852 [107] for describing partially polarized waves by observable power terms and not by amplitudes (and phases).

2.5.1 The Stokes vector for the completely polarized wave

For a monochromatic wave, in the linear {H V} basis, the four Stokes parameters are

$$\begin{aligned} q_0 &= |E_H|^2 + |E_V|^2 \\ q_1 &= |E_H|^2 - |E_V|^2 \\ q_2 &= 2|E_H||E_V|\cos\phi_{HV} \\ q_3 &= 2|E_H||E_V|\sin\phi_{HV} \end{aligned} \tag{2.44}$$

For a completely polarized wave, there are only three independent parameters, which are related as follows

$$q_0^2 = q_1^2 + q_2^2 + q_3^2 \tag{2.45}$$

The Stokes parameters are sufficient to characterize the magnitude and the relative phase, and hence the polarization of a wave. The Stokes parameter q_0 is always equal to the total power (density) of the wave; q_1 is equal to the power in the linear horizontal or vertical polarized components; q_2 is equal to the power in the linearly polarized components at tilt angles $\psi = 45^\circ$ or 135°; and q_3 is equal to the power in the left-handed and right-handed circular polarized components. If any of the parameters q_0, q_1, q_2 or q_3 has a non-zero value, it indicates the presence of a polarized component in the plane wave. The Stokes parameters are also related to the geometric parameters A, χ, and ψ of the polarization ellipse

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} |E_H|^2 + |E_V|^2 \\ |E_H|^2 - |E_V|^2 \\ 2|E_H||E_V|\cos\phi_{HV} \\ 2|E_H||E_V|\sin\phi_{HV} \end{bmatrix} = \begin{bmatrix} A^2 \\ A^2\cos 2\psi \cos 2\chi \\ A^2 \sin 2\psi \cos 2\chi \\ A^2 \sin 2\chi \end{bmatrix} \tag{2.46}$$

which for the normalized case $q_0^2 = e^2 = e_H^2 + e_V^2 = 1$ and

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{\sqrt{|E_H|^2 + |E_V|^2}} \begin{bmatrix} |E_H|^2 + |E_V|^2 \\ |E_H|^2 - |E_V|^2 \\ 2\,\mathrm{Re}\{E_H^* E_V\} \\ 2\,\mathrm{Im}\{E_H^* E_V\} \end{bmatrix} = \begin{bmatrix} e_H^2 + e_V^2 \\ e_H^2 - e_V^2 \\ 2e_H e_V \cos\phi \\ 2e_H e_V \sin\phi \end{bmatrix} = \begin{bmatrix} e^2 \\ e^2 \cos 2\psi \cos 2\chi \\ e^2 \sin 2\psi \cos 2\chi \\ e^2 \sin 2\chi \end{bmatrix} \tag{2.47}$$

2.5.2 The Stokes vector for the partially polarized wave

The Stokes parameter presentation [34] possesses two main advantages in that all of the four parameters are measured as intensities, a crucial fact in optical polarimetry, and the ability to present partially polarized waves in terms of the 2x2 complex hermitian positive semi-definite wave coherency matrix $[J]$ also called the Wolf's coherence matrix [34], defined as:

$$[J] = \left\langle \mathbf{E}\mathbf{E}^{\dagger} \right\rangle = \begin{bmatrix} \left\langle E_H E_H^* \right\rangle & \left\langle E_H E_V^* \right\rangle \\ \left\langle E_V E_H^* \right\rangle & \left\langle E_V E_V^* \right\rangle \end{bmatrix} = \begin{bmatrix} J_{HH} & J_{HV} \\ J_{VH} & J_{VV} \end{bmatrix} = \begin{bmatrix} q_0 + q_1 & q_2 + jq_3 \\ q_2 - jq_3 & q_0 - q_1 \end{bmatrix} \tag{2.48}$$

where $\langle \ldots \rangle = \lim_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} \langle \ldots \rangle dt \right]$ indicating temporal or ensemble averaging assuming stationarity of the wave. We can associate the Stokes vector $\mathbf{q}$ with the coherency matrix $[J]$

$$\begin{aligned} q_0 &= |E_H|^2 + |E_V|^2 = \left\langle E_H E_H^* \right\rangle + \left\langle E_V E_V^* \right\rangle = J_{HH} + J_{VV} \\ q_1 &= |E_H|^2 - |E_V|^2 = \left\langle E_H E_H^* \right\rangle - \left\langle E_V E_V^* \right\rangle = J_{HH} - J_{VV} \\ q_2 &= 2|E_H||E_V|\cos\phi_{HV} = \left\langle E_H E_V^* \right\rangle + \left\langle E_V E_H^* \right\rangle = J_{HV} + J_{VH} \\ q_3 &= 2|E_H||E_V|\sin\phi_{HV} = j\left\langle E_H E_V^* \right\rangle - j\left\langle E_V E_H^* \right\rangle = jJ_{HV} - jJ_{VH} \end{aligned} \tag{2.49}$$

and since $[J]$ is positive semidefinite matrix

$$Det\{[J]\} \geq 0 \quad or \quad q_0^2 \geq q_1^2 + q_2^2 + q_3^2 \tag{2.50}$$

the diagonal elements presenting the intensities, the off-diagonal elements the complex cross-correlation between E_H and E_V, and the $Trace\{[J]\}$, representing the total energy of the wave. For $J_{HV} = 0$ no correlation between E_H and E_V exists, $[J]$ is diagonal with $J_{HH} = J_{VV}$, (i.e. the wave is unpolarized or completely depolarized, and possesses *one degree of freedom only : amplitude*). Whereas, for $Det\{[J]\} = 0$ we find that $J_{VH} J_{HV} = J_{HH} J_{VV}$, and the correlation between E_H and E_V is maximum, and the wave is completely polarized in which case the wave *possesses three degrees of freedom: amplitude, orientation, and ellipticity of the polarization ellipse*. Between these two extreme cases lies the general case of partial polarization, where $Det\{[J]\} > 0$ is indicating a certain degree of statistical dependence between E_H and E_V which can be expressed in terms of the '*degree of coherency*' μ and the 'degree *of polarization*' D_p as

$$\mu_{HV} = |\mu_{HV}| \exp(j\beta_{HV}) = \frac{J_{HV}}{\sqrt{J_{HH} J_{VV}}} \tag{2.51}$$

$$D_p = \left(1 - \frac{4Det\{[J]\}}{\left(Trace\{[J]\}\right)^2} \right)^{1/2} = \frac{\left(q_1^2 + q_2^2 + q_3^2\right)^{1/2}}{q_0} \tag{2.52}$$

where $\mu = D_p = 0$ for totally depolarized and $\mu = D_p = 1$ for fully polarized waves, respectively. However, under a change of polarization basis the elements of the wave coherency matrix $[J]$ depend on the choice of the polarization basis, where according to [52, 53], $[J]$ transforms through a unitary similarity transformation as

$$\left\langle [J_{ij}] \right\rangle = [U_2] \left\langle [J_{mn}] \right\rangle [U_2]^{\dagger} \tag{2.53}$$

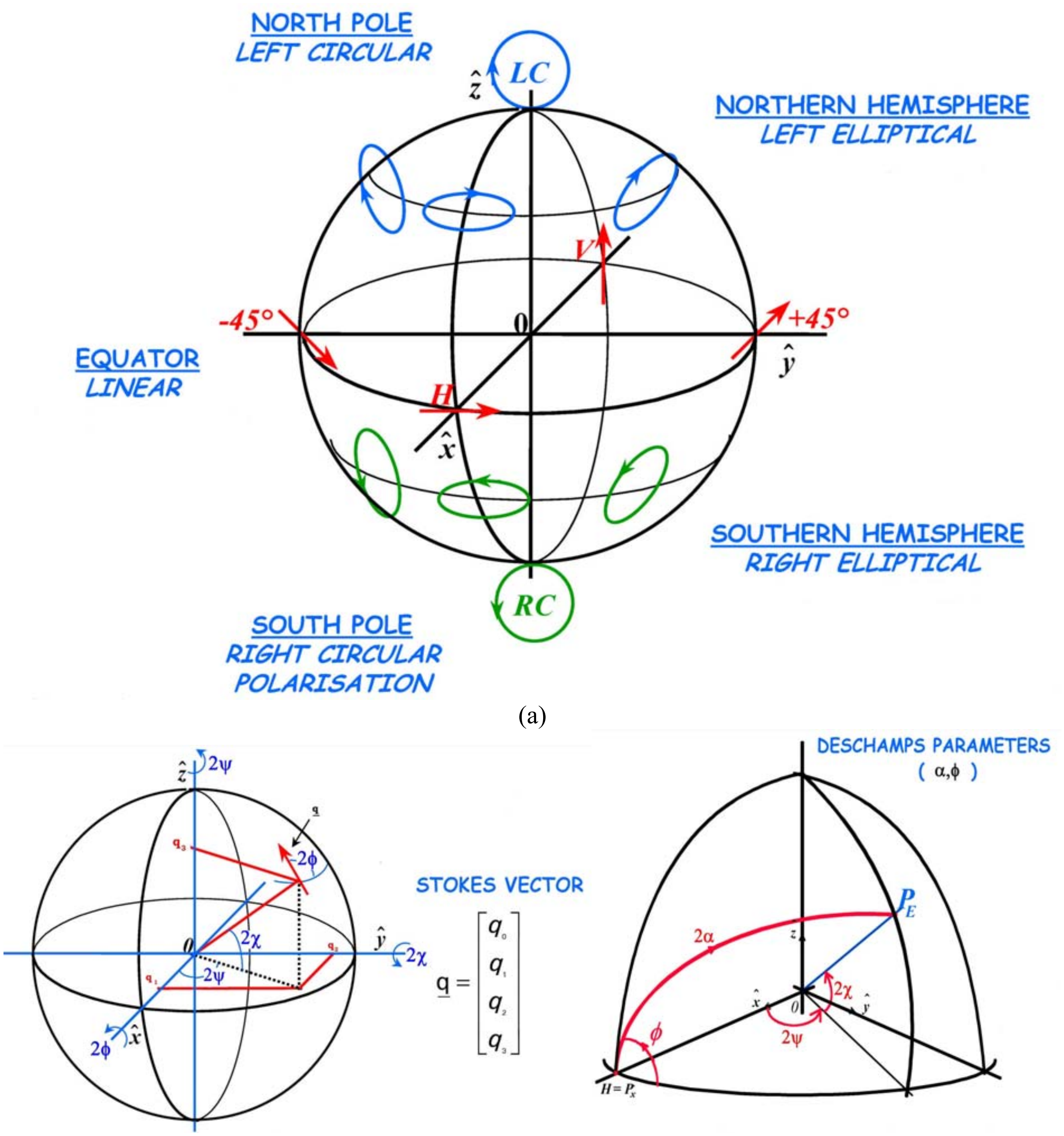
The fact that the trace and the determinant of a hermitian matrix are invariant under unitary similarity transformations means that both, the degree of polarization as well as the total wave intensity are not affected by polarimetric basis transformations. Also, note that the degree of coherence μ_{mn} does depend on the polarization basis. Table 2.2 gives the Jones vector $\mathbf{E}$, Coherency Matrix $[J]$, and Stokes Vector $\mathbf{q}$ for special cases of purely monochromatic wave fields in specific states of polarization.

TABLE 2.2
JONES VECTOR $\mathbf{E}$, COHERENCY MATRIX $[J]$, AND STOKES VECTOR $\mathbf{q}$ FOR SOME STATES OF POLARIZATION

POLARIZATION	{H V} BASIS		
	$\mathbf{E}$	$[J]$	$\mathbf{q}$
Linear Horizontal	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$
Linear Vertical	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$
45° Linear	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$
135° Linear	$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$
Left-handed Circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-j\\j&1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$
Right-handed Circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&j\\-j&1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$

2.6 The Poincaré Polarization Sphere

The Poincaré sphere, shown in Fig. 2.4 for the representation of wave polarization using the Stokes vector and the Deschamps parameters (α,ϕ) is a useful graphical aid for the visualization of polarization effects. There is one-to-one correspondence between all possible polarization states and points on the Poincaré sphere, with the linear polarizations mapped onto the equatorial plane (x = 0) with the *'zenith'* presenting left-handed circular and the *'nadir'* right-handed circular polarization states according to the IEEE standard notation $\exp(+j\omega t)$ [102], and any set of orthogonally fully polarized polarization states being mapped into antipodal points on the Poincaré sphere [108].



(b) (c)

Fig. 2.4 Poincaré Sphere Representations (Courtesy of Prof. E. Pottier)

2.6.1 The polarization state on the Poincaré sphere for the {H V} basis

In the Poincaré sphere representation, the polarization state is described by a point P on the sphere, where the three Cartesian coordinate components are q_1, q_2, and q_3 according to (2.46). So, for any state of a completely polarized wave, there corresponds one point $P(q_1, q_2, q_3)$ on the sphere of radius q_0, and vice versa. In Fig. 2.5, we can see that the longitude and latitude of the point P are related to the geometric parameter of the polarization ellipse and they are 2ψ and 2χ respectively.

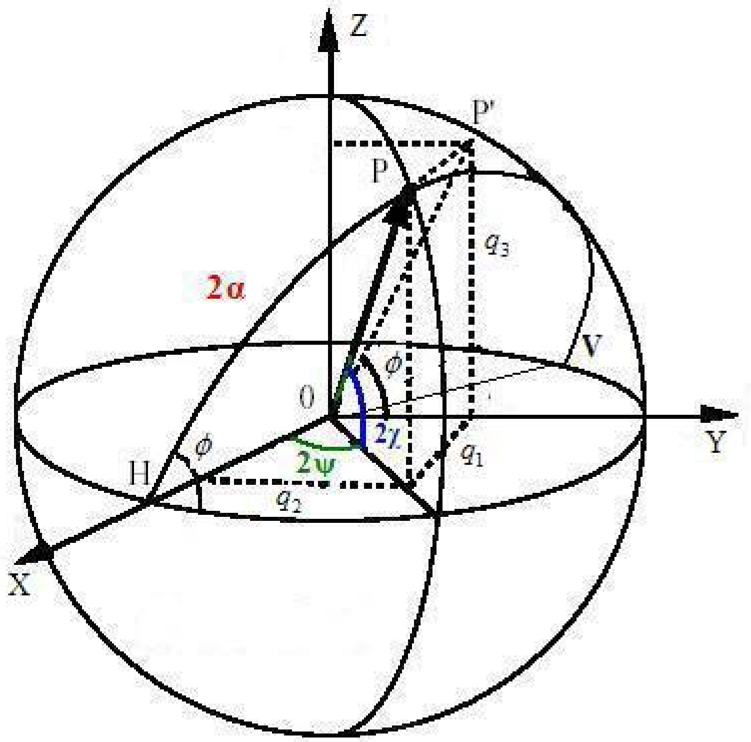


Fig. 2.5 The Poincaré sphere and the parameters α_{HV} and ϕ_{HV}

In addition, the point P on the Poincaré sphere can also be represented by the angles α_{HV} and ϕ_{HV}. From (2.37) and (2.46) we find that

$$\frac{q_1}{q_0} = \cos 2\psi \cos 2\chi = \cos 2\alpha_{HV} \tag{2.54}$$

where $\cos 2\alpha_{HV}$ is the direction cosine of the Stokes vector $\mathbf{q}$ with respect to the X-axis, i.e., the angle $2\alpha_{HV}$ is the angle between $\mathbf{q}$ and the X-axis. The angle ϕ_{HV} is the angle between the equator and the great circle with basis diameter HV through the point P, and it is equal to the angle between the XOY plane and the XOP plane. Drawing a projecting line from point P to the YOZ plane, the intersecting point P' is on the XOP plane, so $\phi_{HV} = \angle \mathrm{YOP}'$ ($\phi_{HV} = \phi$ in Fig. 2.5). On the YOZ plane we find that

$$\tan\phi_{HV} = \tan\angle \mathrm{YOP}' = \frac{q_3}{q_2} = \frac{\tan 2\chi}{\sin 2\psi} \tag{2.55}$$

which satisfies equations (2.46) and (2.37).

2.6.2 The polarization ratio on the Poincaré sphere for different polarization bases

Also, it can be shown that a polarization state can be represented in different polarization bases. Any polarization basis consists of two unit vectors which are located at two corresponding antipodal points on the Poincaré sphere. Fig 2.6 shows how the polarization state P on the Poincaré sphere can be represented in three polarization bases, {H V}, {45° 135°}, and {L R}. The complex polarization ratios are given by

$$\rho_{HV} = |\rho_{HV}| \exp(j\phi_{HV}) = \begin{cases} \tan\alpha_{HV} \exp(j\phi_{HV}) & 0 < \alpha_{HV} < \dfrac{\pi}{2} \\ -\tan\alpha_{HV} \exp(j\phi_{HV}) & \dfrac{\pi}{2} < \alpha_{HV} < \pi \end{cases} \tag{2.56}$$

$$\rho_{45^\circ 135^\circ} = |\rho_{45^\circ 135^\circ}| \exp(j\phi_{45^\circ 135^\circ}) = \begin{cases} \tan\alpha_{45^\circ 135^\circ} \exp(j\phi_{45^\circ 135^\circ}) & 0 < \alpha_{45^\circ 135^\circ} < \dfrac{\pi}{2} \\ -\tan\alpha_{45^\circ 135^\circ} \exp(j\phi_{45^\circ 135^\circ}) & \dfrac{\pi}{2} < \alpha_{45^\circ 135^\circ} < \pi \end{cases} \tag{2.57}$$

$$\rho_{LR} = |\rho_{LR}| \exp(j\phi_{LR}) = \begin{cases} \tan\alpha_{LR} \exp(j\phi_{LR}) & 0 < \alpha_{LR} < \dfrac{\pi}{2} \\ -\tan\alpha_{LR} \exp(j\phi_{LR}) & \dfrac{\pi}{2} < \alpha_{LR} < \pi \end{cases} \tag{2.58}$$

where $\tan\alpha_{HV}$, $\tan\alpha_{45^\circ 135^\circ}$, and $\tan\alpha_{LR}$ are the ratios of the magnitudes of the corresponding orthogonal components, and ϕ_{HV}, $\phi_{45^\circ 135^\circ}$, and ϕ_{LR} are the phase differences between the corresponding orthogonal components

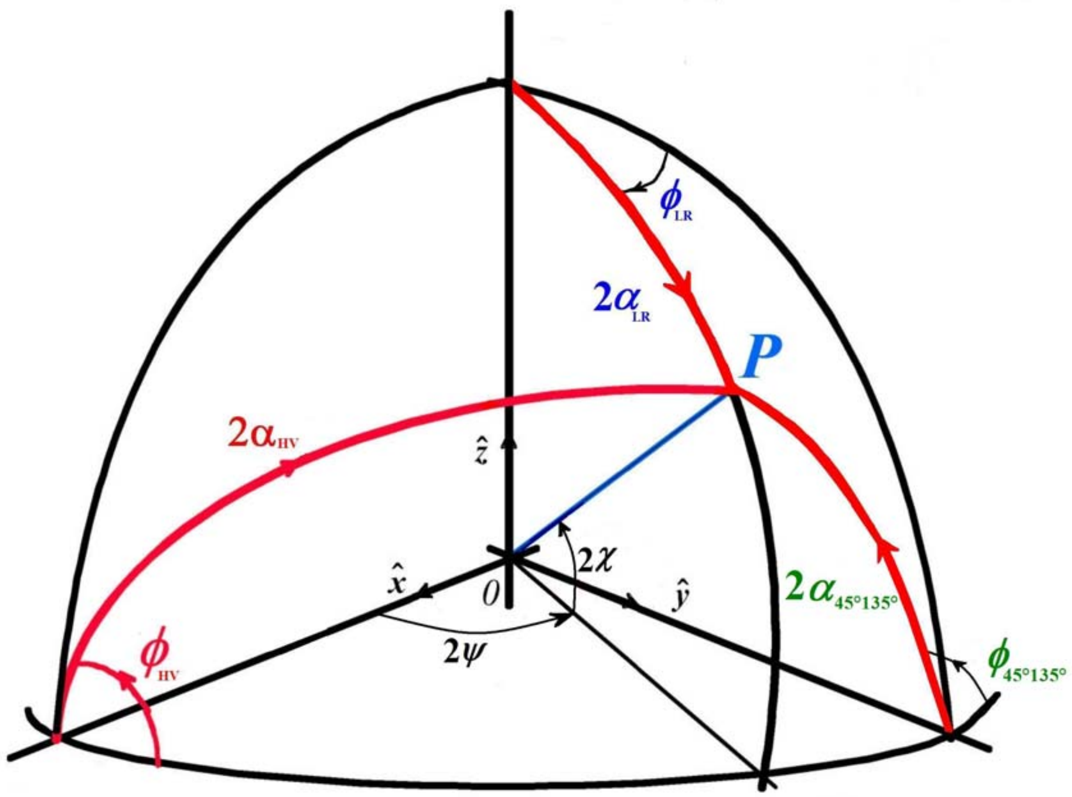


Fig 2.6 The Polarization State P in Different Polarization Bases

2.6.3 The relationship between the Stokes vector and the polarization ratio for different polarization bases

First, consider the polarization ratio ρ_{HV} defined in the {H V} basis. Because $\cos 2\alpha_{HV}$ is the direction cosine of the Stokes vector $\mathbf{q}$ with respect to the X-axis, we find

$$\frac{q_1}{q_0} = \cos 2\alpha_{HV} = \frac{1-\tan^2\alpha_{HV}}{1+\tan^2\alpha_{HV}} = \frac{1-|\rho_{HV}|^2}{1+|\rho_{HV}|^2} \tag{2.59}$$

the straight forward solution for $|\rho_{HV}|$ is

$$|\rho_{HV}| = \sqrt{\frac{q_0 - q_1}{q_0 + q_1}} \tag{2.60}$$

from (2.54), we find

$$\phi_{HV} = \angle \mathrm{YOP'} = \tan^{-1}\left(\frac{q_3}{q_2}\right) \tag{2.61}$$

Combining above two equations yields

$$\rho_{HV} = |\rho_{HV}|\exp(j\phi_{HV}) = \sqrt{\frac{q_0 - q_1}{q_0 + q_1}}\exp\left\{ j\tan^{-1}\left(\frac{q_3}{q_2}\right)\right\} \tag{2.62}$$

For a completely polarized wave, we may obtain the Stokes vector in terms of the polarization ratio ρ_{HV} by applying

$$\begin{aligned}
q_0 &= \sqrt{q_1^2 + q_2^2 + q_3^2} = 1 \\
q_1 &= \frac{1-|\rho_{HV}|^2}{1+|\rho_{HV}|^2} = \cos 2\alpha_{HV} \\
q_2 &= \frac{2|\rho_{HV}|\cos\phi_{HV}}{1+|\rho_{HV}|^2} = \frac{2\tan\alpha_{HV}\cos\phi_{HV}}{1+|\tan\alpha_{HV}|^2} = \sin(2\alpha_{HV})\cos\phi_{HV} \\
q_3 &= \frac{2|\rho_{HV}|\sin\phi_{HV}}{1+|\rho_{HV}|^2} = \sin(2\alpha_{HV})\sin\phi_{HV}
\end{aligned} \tag{2.63}$$

The sign of the three components of the Stokes vector is summarized in Table 2.3.

Secondly, consider the polarization ratio $\rho_{45°135°}$ defined in the {45° 135°} basis. The $\cos 2\alpha_{45°135°}$ is the direction cosine of the Stokes vector $\mathbf{q}$ with respect to the Y-axis. So similarly, with

$$
\begin{aligned}
q_0 &= 1 \\
\left|\rho_{45^\circ 135^\circ}\right| &= \sqrt{\frac{q_0 - q_2}{q_0 + q_2}} \\
\phi_{45^\circ 135^\circ} &= \tan^{-1}\left(-\frac{q_3}{q_1}\right)
\end{aligned}
\tag{2.64}
$$

TABLE 2.3

THE SIGN OF THE q_1, q_2, AND q_3 PARAMETERS IN THE {H V} BASIS

ϕ_{HV}	α_{HV}	q_1	q_2	q_3
$0 < \phi_{HV} < \frac{\pi}{2}$	$0 < 2\alpha_{HV} < \frac{\pi}{2}$	+	+	+
	$\frac{\pi}{2} < 2\alpha_{HV} < \pi$	−	+	+
	$0 < 2\alpha_{HV} < \frac{3\pi}{2}$	−	−	−
	$\frac{3\pi}{2} < 2\alpha_{HV} < 2\pi$	+	−	−
$-\frac{\pi}{2} < \phi_{HV} < 0$	$0 < 2\alpha_{HV} < \frac{\pi}{2}$	+	+	−
	$\frac{\pi}{2} < 2\alpha_{HV} < \pi$	−	+	−
	$0 < 2\alpha_{HV} < \frac{3\pi}{2}$	−	−	+
	$\frac{3\pi}{2} < 2\alpha_{HV} < 2\pi$	+	−	+

Then the polarization ratio $\rho_{45^\circ 135^\circ}$ can be determined by the Stokes vector $\mathbf{q}$

$$
\rho_{45^\circ 135^\circ} = \sqrt{\frac{q_0 - q_2}{q_0 + q_2}} \exp\left\{ j \tan^{-1}\left(-\frac{q_3}{q_1}\right)\right\}
\tag{2.65}
$$

Also, the Stokes vector $\mathbf{q}$ can be determined by the polarization ratio $\rho_{45^\circ 135^\circ}$ as follows:

$$
\begin{aligned}
q_0 &= 1 \\
q_1 &= \frac{2\left|\rho_{45^\circ 135^\circ}\right| \cos\phi_{45^\circ 135^\circ}}{1 + \left|\rho_{45^\circ 135^\circ}\right|^2} = -\sin 2\alpha_{45^\circ 135^\circ} \cos\phi_{45^\circ 135^\circ} \\
q_2 &= \frac{1 - \left|\rho_{45^\circ 135^\circ}\right|^2}{1 + \left|\rho_{45^\circ 135^\circ}\right|^2} = \cos 2\alpha_{45^\circ 135^\circ} \\
q_3 &= \frac{2\left|\rho_{45^\circ 135^\circ}\right| \sin\phi_{45^\circ 135^\circ}}{1 + \left|\rho_{45^\circ 135^\circ}\right|^2} = \sin 2\alpha_{45^\circ 135^\circ} \sin\phi_{45^\circ 135^\circ}
\end{aligned}
\tag{2.66}
$$

Finally, consider the polarization ratio ρ_{LR} defined in the {L R} basis. Similarly, because the $\cos 2\alpha_{LR}$ is the direction cosine of the Stokes vector $\mathbf{q}$ with respect to the Z-axis, the polarization ratio ρ_{LR} can be determined by the Stokes vector $\mathbf{q}$ as:

$$\rho_{LR} = \sqrt{\frac{q_0 - q_3}{q_0 + q_3}} \exp\left\{ j \tan^{-1}\left(\frac{q_2}{q_1}\right)\right\} \tag{2.67}$$

Inversely,

$$\begin{aligned} q_0 &= 1 \\ q_1 &= \frac{2|\rho_{LR}|\cos\phi_{LR}}{1+|\rho_{LR}|^2} = \sin 2\alpha_{LR}\cos\phi_{LR} \\ q_2 &= \frac{2|\rho_{LR}|\sin\phi_{LR}}{1+|\rho_{LR}|^2} = \sin 2\alpha_{LR}\sin\phi_{LR} \\ q_3 &= \frac{1-|\rho_{LR}|^2}{1+|\rho_{LR}|^2} = \cos 2\alpha_{LR} \end{aligned} \tag{2.68}$$

TABLE 2.4
ALTERNATE EXPRESSIONS FOR NORMALIZED STOKES VECTOR PRESENTATIONS ON THE POLARIZATION SPHERE

	χ,ψ	α_{HV},ϕ_{HV}	$\alpha_{45^\circ 135^\circ},\phi_{45^\circ 135^\circ}$	α_{LR},ϕ_{LR}
q_0	1	1	1	1
q_1	$\cos 2\chi \cos 2\psi$	$\cos 2\alpha_{HV}$	$-\sin 2\alpha_{45^\circ 135^\circ}\cos\phi_{45^\circ 135^\circ}$	$\sin 2\alpha_{LR}\cos\phi_{LR}$
q_2	$\cos 2\chi \sin 2\psi$	$\sin 2\alpha_{HV}\cos\phi_{HV}$	$\cos 2\alpha_{45^\circ 135^\circ}$	$\sin 2\alpha_{LR}\sin\phi_{LR}$
q_3	$\sin 2\chi$	$\sin 2\alpha_{HV}\sin\phi_{HV}$	$\sin 2\alpha_{45^\circ 135^\circ}\sin\phi_{45^\circ 135^\circ}$	$\cos 2\alpha_{LR}$

2.6.4 The Poincaré polarization sphere and complex polarization ratio plane

Using the Riemann transformation, Poincaré introduced the polarization sphere representation of Fig. 2.5 which gives a relationship between the polarization ratio ρ and its corresponding spherical coordinates on the Poincaré sphere. First we need to introduce an auxiliary complex parameter $\mathrm{u}(\rho)$, which is defined by the Riemann transformation [14] of the surface of the sphere onto the polar grid as follows,

$$\mathrm{u}(\rho) = \frac{1 - j\rho}{1 + j\rho} \tag{2.69}$$

in the {H V} basis, $\rho_{HV} = \tan\alpha_{HV}\exp\{j\phi_{HV}\} = \tan\alpha_{HV}(\cos\phi_{HV} + j\sin\phi_{HV})$, then

$$\mathrm{u} = \frac{(1+\tan\alpha_{HV}\sin\phi_{HV}) - j\tan\alpha_{HV}\cos\phi_{HV}}{(1-\tan\alpha_{HV}\sin\phi_{HV}) + j\tan\alpha_{HV}\cos\phi_{HV}}$$

$$|\mathrm{u}|^2 = \frac{1+2\tan\alpha_{HV}\sin\phi_{HV}) + \tan^2\alpha_{HV}}{1-2\tan\alpha_{HV}\sin\phi_{HV}) + \tan^2\alpha_{HV}}$$

$$\frac{|u|^2-1}{|u|^2+1}=\frac{2\tan\alpha_{HV}}{1+\tan^2\alpha_{HV}}\sin\phi_{HV}=\sin 2\alpha_{HV}\sin\phi_{HV}$$

according to (2.36) and Fig. 2.4b, the polar angle $\Theta=\pi/2-2\chi$ can be obtained from

$$\frac{|u|^2-1}{|u|^2+1}=\sin 2\chi=\sin(\pi/2-\Theta)=\cos\Theta$$

so that

$$\Theta=\cos^{-1}\left(\frac{|u|^2-1}{|u|^2+1}\right) \tag{2.70}$$

also, according to (2.36) and Fig. 2.4b, the spherical azimuthal angle $\Phi=2\psi$ can be obtained from $-\frac{\mathrm{Im}\{u\}}{\mathrm{Re}\{u\}}=\frac{2\tan\alpha_{HV}\cos\phi_{HV}}{1-\tan^2\alpha_{HV}}=\tan 2\psi=\tan\Phi$, so that the spherical azimuthal angle Φ becomes

$$\Phi=\tan^{-1}\left(\frac{\mathrm{Im}\{u\}}{\mathrm{Re}\{u\}}\right) \tag{2.71}$$

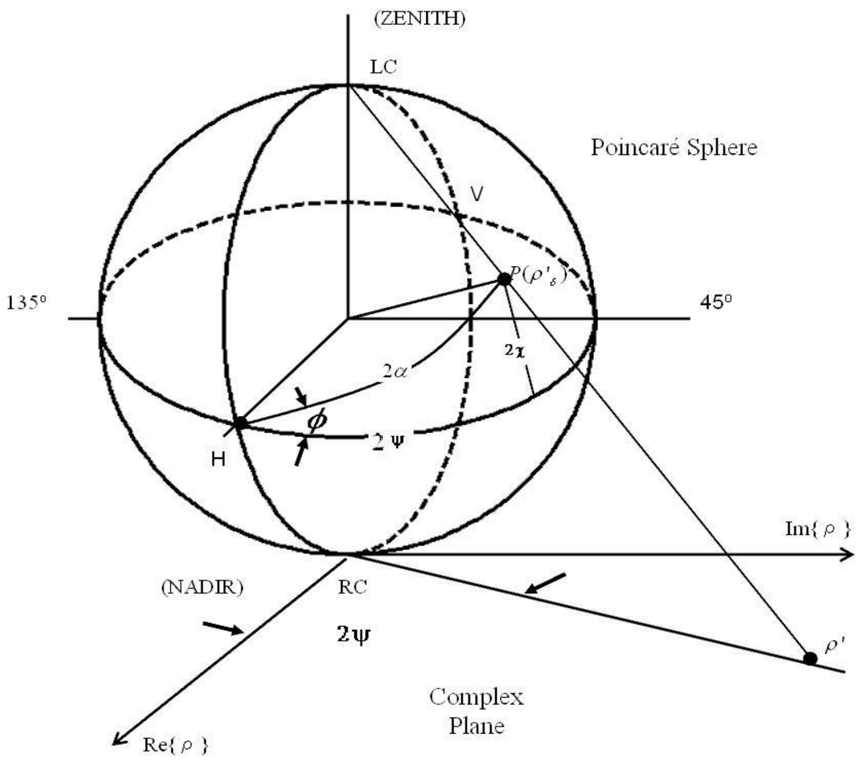


Fig. 2.7 Poincaré Sphere and the Complex Plane

2.7 Wave Decomposition Theorems

The diagonalization of $\left[J_{ij}\right]$, under the unitary similarity transformation is equivalent to finding an orthonormal polarization basis in which the coherency matrix is diagonal or

$$\begin{bmatrix} J_{mm} & J_{mn} \\ J_{nm} & J_{nn} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} e_{11}^* & e_{12}^* \\ e_{21}^* & e_{22}^* \end{bmatrix} \tag{2.72}$$

where λ_1 and λ_2 are the real non-negative eigenvalues of $[J]$ with $\lambda_1 \geq \lambda_2 \geq 0$, and $\hat{e}_1 = \begin{bmatrix} e_{11} & e_{12} \end{bmatrix}^T$ and $\hat{e}_2 = \begin{bmatrix} e_{21} & e_{22} \end{bmatrix}^T$ are the complex orthogonal eigenvectors which define $[U_2]$ and a polarization basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$ in which $[J]$ is diagonal. $[J]$ is Hermitian and hence normal and every normal matrix can be unitarily diagonalized . Being positive semidefinite the eigenvalues are nonnegative.

2.8 The Wave Dichotomy of Partially Polarized Waves

The solution of (2.72) provides two equivalent interpretations of partially polarized waves [28]: i) separation into fully polarized $[J_1]$, and into completely depolarized $[J_2]$ components

$$[J] = (\lambda_1 - \lambda_2)[J_1] + \lambda_2 [I_2] \tag{2.72}$$

where $[I_2]$ is the 2x2 identity matrix ; ii) non-coherency of two orthogonal completely polarized wave states represented by the eigenvectors and weighed by their corresponding eigenvalues as

$$[J] = (\lambda_1)[J_1] + \lambda_2 [J_2] = \lambda_1 (\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1^\dagger) + \lambda_2 (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2^\dagger) \tag{2.74}$$

where $Det\{[J_1]\} = Det\{[J_2]\} = 0$; and if $\lambda_1 = \lambda_2$ the wave is totally depolarized (degenerate case) whereas for $\lambda_2 = 0$, the wave is completely polarized. Both models are unique in the sense that no other decomposition in form of a separation of two completely polarized waves or of a completely polarized with noise is possible for a given coherency matrix, which may be reformulated in terms of the '*degree of polarization'* D_p as

$$D_p = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}, \;\; 0 \;\; (\lambda_1 = \lambda_2) \;\; and \;\; 1 \;\; (\lambda_2 = 0) \tag{2.75}$$

for a partially unpolarized and completely polarized wave. The fact that the eigenvalues λ_1 and λ_2 are invariant under polarization basis transformation makes D_p an important basis-independent parameter.

2.9 Polarimetric Wave Entropy

Alternately to the degrees of wave coherency μ and polarization D_p, the polarimetric wave entropy H_ω [28] provides another measure of the correlated wave structure of the coherency matrix $[J]$, where by using the logarithmic sum of eigenvalues

$$H_\omega = \sum_{i=1}^{2} \{-P_i \log_2 P_i\} \qquad with \;\; P_i = \frac{\lambda_i}{\lambda_1 + \lambda_2} \tag{2.76}$$

so that $P_1 + P_2 = 1$ and the normalized wave entropy ranges from $0 \leq H_\omega \leq 1$ where for a completely polarized wave with $\lambda_2 = 0$ and $H_\omega = 0$, while a completely randomly polarized wave with $\lambda_1 = \lambda_2$ possesses maximum entropy $H_\omega = 1$.

2.10 Alternate Formulations of the Polarization Properties of Electromagnetic Vector Waves

There exist several alternate formulations of the polarization properties of electromagnetic vector waves including; (i) the '*Four-vector Hamiltonian'* formulation frequently utilized by Zhivotovsky [109] and by Czyz [110], which may be useful in a more concise description of partially polarized waves ; (ii) the '*Spinorial formulation'* used by Bebbington [32], and in general gravitation theory [111] ; and (iii) a pseudo-spinorial formulation by Czyz [110] is in development which are most essential tools for describing the general bi-static (non-symmetric) scattering matrix cases for both the coherent (*3-D Poincaré sphere and the 3-D polarization spheroid*) and the partially polarized (*4-D Zhivotovsky sphere and 4-D spheroid)* cases [109]. Because of the exorbitant excessive additional mathematical tools required, and not commonly accessible to engineers and applied scientists, these formulations are not presented here but deserve our fullest attention in future analyses.

3. The Electromagnetic Vector Scattering Operator and the Polarimetric Scattering Matrices

The electromagnetic vector wave interrogation with material media is described by the Scattering Operator $\left[S(\mathbf{k}_s/\mathbf{k}_i)\right]$ with $\mathbf{k}_s,\mathbf{k}_i$ representing the wave vectors of the scattered and incident, $\mathbf{E}^s(\mathbf{r}),\mathbf{E}^i(\mathbf{r})$ respectively, where

$$\mathbf{E}^s(\mathbf{r}) = E_0^s \exp(-j\mathbf{k}_s\cdot\mathbf{r}) = \mathbf{e}_s E_0^s \exp(-j\mathbf{k}_s\cdot\mathbf{r}) \tag{3.1}$$

is related to

$$\mathbf{E}^i(\mathbf{r}) = E_0^i \exp(-j\mathbf{k}_i\cdot\mathbf{r}) = \mathbf{e}_i E_0^i \exp(-j\mathbf{k}_i\cdot\mathbf{r}) \tag{3.2}$$

$$\mathbf{E}^s(\mathbf{r}) = \frac{\exp(-j\mathbf{k}_s\cdot\mathbf{r})}{r}\left[S(\mathbf{k}_s/\mathbf{k}_i)\right]\mathbf{E}^i(\mathbf{r}) \tag{3.3}$$

The scattering operator $\left[S(\mathbf{k}_s/\mathbf{k}_i)\right]$ is obtained from rigorous application of vector scattering and diffraction theory, to the specific scattering scenario under investigation which is not further discussed here, but we refer to [97] for a thought-provoking formulation of these still open problems.

3.1 The Scattering Scenario and the Scattering Coordinate Framework

The scattering operator $\left[S(\mathbf{k}_s/\mathbf{k}_i)\right]$ appears as the output of the scattering process for an arbitrary input $\mathbf{E}_0^i$, which must carefully be defined in terms of the scattering scenario; and, its proper unique formulation is of intrinsic importance to both optical and radar polarimetry. Whereas in optical remote sensing mainly the '*forward scattering through translucent media'* is considered, in radar remote sensing the '*back scattering from distant, opaque open and closed surfaces*' is of interest, where in radar target backscattering we usually deal with closed surfaces whereas in SAR imaging one deals with open surfaces. These two distinct cases of optical versus radar scattering are treated separately using two different reference frames; the '*Forward (anti-monostatic) Scattering Alignment (FSA)*' versus the '*Back Bistatic Scattering Alignment (BSA)*' from which the '*Monostatic Arrangement*' is derived as shown in Fig. 3.1. In the following, separately detailed for both the FSA and BSA systems are shown in Figs. 3.2 and 3.3.

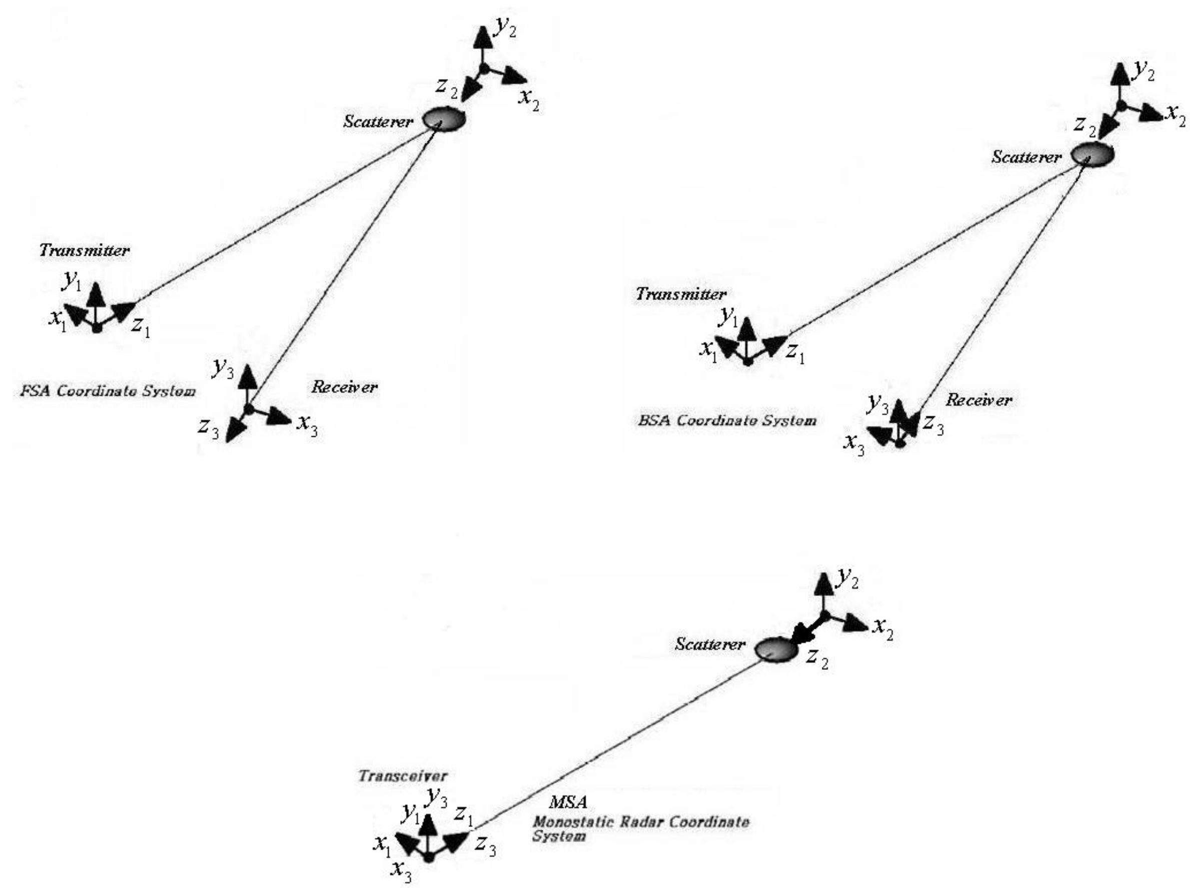
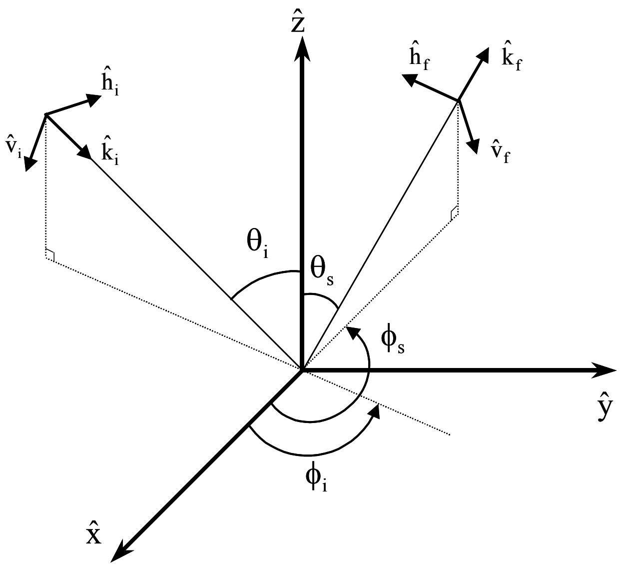
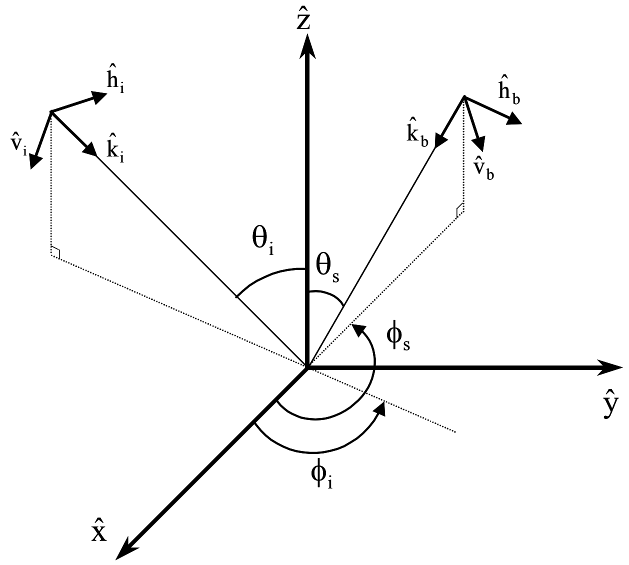


Fig. 3.1 Comparison of the FSA, BSA, and MSA Coordinate Systems



$$\hat{k}_f = \sin\theta_s \cos\phi_s \hat{x} + \sin\theta_s \sin\phi_s \hat{y} + \cos\theta_s \hat{z}$$
$$\hat{v}_f = \cos\theta_s \cos\phi_s \hat{x} + \cos\theta_s \sin\phi_s \hat{y} - \sin\theta_s \hat{z}$$
$$\hat{h}_f = -\sin\phi_s \hat{x} + \cos\phi_s \hat{y}$$

Fig. 3.2 Detailed Forward Scattering Alignment (FSA)



$$\hat{k}_b = -\sin\theta_s \cos\phi_s \hat{x} - \sin\theta_s \sin\phi_s \hat{y} - \cos\theta_s \hat{z}$$
$$\hat{v}_b = \cos\theta_s \cos\phi_s \hat{x} + \cos\theta_s \sin\phi_s \hat{y} - \sin\theta_s \hat{z}$$
$$\hat{h}_b = \sin\phi_s \hat{x} - \cos\phi_s \hat{y}$$

Fig. 3.3 Detailed Back Scattering Alignment (BSA)

3.2 The 2x2 Jones Forward [*J*] versus 2x2 Sinclair [*S*] Back-Scattering Matrices

Since we are dealing here with radar polarimetry, interferometry and polarimetric interferometry, the '*bistatic BSA reference frame'* is more relevant and is here introduced only for reasons of brevity but dealing both with the bistatic and the monostatic cases ; where we refer to [52, 53], [76] and [19] for a full treatment of the *'anti-monostatic FSA reference frame'*. Here, we refer to the dissertation of Papathanassiou [97], the textbook of Mott [76], and meticulous derivations of Lüneburg [52] for more detailed treatments of the subject matter, but we follow here the derivation presented in [19]. Using the coordinates of Fig. 3.1 with right-handed coordinate systems; $x_1\, y_1\, z_1$, $x_2\, y_2\, z_2$, $x_3\, y_3\, z_3$; denoting the transmitter, scatterer and receiver coordinates, respectively, a wave incident on the scatterer from the transmitter is given by the transverse components E_{x_1} and E_{y_1} in the right-handed coordinate system $x_1\, y_1\, z_1$ with the z_1 axis pointed at the target. The scatterer coordinate system $x_2\, y_2\, z_2$ is right-handed with z_2 pointing away from the scatterer toward a receiver. BSA Coordinate System $x_3\, y_3\, z_3$ is right-handed with z_3 pointing toward the scatterer. It would coincide with the transmitter system $x_1\, y_1\, z_1$ if the transmitter and receiver were co-located. The wave reflected by the target to the receiver may be described in either the transverse components E_{x_2} and E_{y_2} or by the reversed components E_{x_3} and E_{y_3}. Both conventions are used, leading to different matrix formulations. The incident and transmitted or reflected (scattered) fields are given by two-component vectors; therefore the relationship between them must be a 2x2 matrix. If the scattered field is expressed in $x_3\, y_3\, z_3$ coordinates (BSA), the fields are related by the Sinclair matrix $\left[S\right]$, thus

$$\begin{bmatrix} E^s_{x_3} \\ E^s_{y_3} \end{bmatrix} = \frac{1}{\sqrt{4\pi r_2}} \begin{bmatrix} S_{x_3 x_1} & S_{x_3 y_1} \\ S_{y_3 x_1} & S_{y_3 y_1} \end{bmatrix} \begin{bmatrix} E^i_{x_1} \\ E^i_{y_1} \end{bmatrix} e^{-j k r_2} \tag{3.4}$$

and if the scattered field is in $x_2\, y_2\, z_2$ coordinates (FSA), it is given by the product of the Jones matrix $\left[J\right]$ with the incident field, thus

$$\begin{bmatrix} E^s_{x_3} \\ E^s_{y_3} \end{bmatrix} = \frac{1}{\sqrt{4\pi r_2}} \begin{bmatrix} T_{x_2x_1} & T_{x_2y_1} \\ T_{y_2x_1} & T_{y_2y_1} \end{bmatrix} \begin{bmatrix} E^i_{x_1} \\ E^i_{y_1} \end{bmatrix} e^{-j\,kr_2} \tag{3.5}$$

In both equations the incident fields are those at the target, the received fields are measured at the receiver, and r_2 is the distance from target to receiver. The *'Sinclair matrix* $[S]$ *'* is mostly used for **back-scattering**, but is readily extended to the **bistatic scattering** case. If the name **scattering matrix** is used without qualification, it normally refers to the Sinclair matrix $[S]$. In the general bistatic scattering case, the elements of the Sinclair matrix are not related to each other, except through the physics of the scatterer. However, if the receiver and transmitter are co-located, as in the **mono-static** or back-scattering situation, and if the medium between target and transmitter is reciprocal, mainly the Sinclair matrix $[S(AB)]$ is symmetric, i.e. $S_{AB} = S_{BA}$. The Jones matrix is used for the forward transmission case; and if the medium between target and transmitter, without Faraday rotation, the Jones matrix is usually normal. However, it should be noted that the Jones matrix is not in general normal, i.e., in general the Jones matrix does not have orthogonal eigenvectors. Even the case of only one eigenvector (and a generalized eigenvector) has been considered in optics (homogeneous and inhomogeneous Jones matrices). If the coordinate systems being used are kept in mind, the numerical subscripts can be dropped.

It is clear that in the bistatic case, the matrix elements for a target depend on the orientation of the target with respect to the line of sight from transmitter to target and on its orientation with respect to the target-receiver line of sight. In the forms (3.4) and (3.5) the matrices are **absolute matrices**, and with their use the phase of the scattered wave can be related to the phase of the transmitted wave, which is strictly required in the case of polarimetric interferometry. If this phase relationship is of no interest, as in the case of mono-static polarimetry, the distinct phase term can be neglected, and one of the matrix elements can be taken as real. The resulting form of the Sinclair matrix is called the **relative scattering matrix**. In general the elements of the scattering matrix are dependent on the frequency of the illuminating wave [19, 14, 15].

Another target matrix parameter that should be familiar to all who are interested in microwave remote sensing is the **radar cross section (RCS)**. It is proportional to the power received by a radar and is the area of an equivalent target that intercepts a power equal to its area multiplied by the power density of an incident wave and re-radiates it equally in all directions to yield a receiver power equal to that produced by the real target. The radar cross section depends on the polarization of both transmitting and receiving antennas. Thus the radar cross section may be specified as HH (horizontal receiving and transmitting antennas), HV (horizontal receiving and vertical transmitting antennas), etc. When considering ground reflections, the cross section is normalized by the size of the ground patch illuminated by the wave from the radar. The cross section is not sufficient to describe the polarimetric behavior of a target. In terms of the Sinclair matrix $[S]$, and the normalized effective lengths of transmitting and receiving antennas, $\hat{\mathbf{h}}_t$ and $\hat{\mathbf{h}}_r$, respectively**,** the radar cross section is

$$\sigma_{rt} = \left| \hat{h}_r^{\,T} \,[S]\, \hat{h}_t \right|^2 \tag{3.6}$$

A polarimetrically correct form of the **radar equation** that specifies received power in terms of antenna and target parameters is

$$W_{rt} = \frac{W_t\, G_t(\theta,\phi)\, A_{er}(\theta,\phi)}{(4\pi\, r_1 r_2)^2} \left| \hat{h}_r^{\,T} \,[S]\, \hat{h}_t \right|^2 \tag{3.7}$$

where W_t is the transmitter power and subscripts t and r identify transmitter and receiver, and its properties are defined in more detail in Mott [76] and in [19]. The effective antenna height $\hat{\mathbf{h}}(\theta, \phi)$, is defined via the electric field $\mathbf{E}^{t}(r, \theta, \phi)$, radiated by an antenna in its far field, as

$$\mathbf{E}^{t}(r, \theta, \phi) = \frac{jZ_0 I}{2\lambda r} \exp(-jkr) \hat{\mathbf{h}}(\theta, \phi), \tag{3.8}$$

with Z_0 the characteristic impedance, λ the wavelength, and I the antenna current.

3.3 Basis Transformations of the 2x2 Sinclair Scattering Matrix [*S*]

Redefining the incident and scattering cases in terms of the standard {H V} notation with $H = x$, $V = y$ and with proper re-normalization, we redefine (3.1) as

$$\mathbf{E}^{s}_{HV} = [S]_{HV} \mathbf{E}^{*}_{HV} \quad \text{or} \quad \mathbf{E}^{s}(HV) = \left[S(HV)\right] \mathbf{E}^{*}(HV) \tag{3.9}$$

where the complex conjugation results from inversion of the coordinate system in the BSA arrangement which invites a more rigorous formulation in terms of directional Sinclair vectors including the concepts of time reversal as treated by Lüneburg [52]. Using these Sinclair vector definitions one can show that the transformation from one orthogonal polarisation basis {H V} into another {i j} or {A B} is a unitary congruence (unitary consimilarity) transformation of the original Sinclair scattering matrix $[S]_{HV}$ into $[S]_{ij}$, where

$$[S]_{ij} = [U_2][S]_{HV}[U_2]^{T} \quad \text{or} \quad \left[S(ij)\right] = \left[U_2\right] \left[S(HV)\right] = \left[U_2\right]^{T} \tag{3.10}$$

with $[U_2]$ given by (2.23), so that the components of the general non-symmetric scattering matrix for the bistatic case in the new polarization basis, characterized by a complex polarization ratio ρ, can be written as [81, 25]

$$S_{ii} = \frac{1}{1+\rho\rho^{*}} \left[S_{HH} - \rho^{*} S_{HV} - \rho^{*} S_{VH} + \rho^{*2} S_{VV}\right] \tag{3.11}$$

$$S_{ij} = \frac{1}{1+\rho\rho^{*}} \left[\rho S_{HH} + S_{HV} - \rho\rho^{*} S_{VH} - \rho^{*} S_{VV}\right]$$

$$S_{ji} = \frac{1}{1+\rho\rho^{*}} \left[\rho S_{HH} - \rho\rho^{*} S_{HV} + S_{VH} - \rho^{*} S_{VV}\right]$$

$$S_{jj} = \frac{1}{1+\rho\rho^{*}} \left[\rho^{2} S_{HH} + \rho S_{HV} - \rho S_{VH} + S_{VV}\right]$$

There exist three invariants for the general bistatic case (BSA) under the change-of-basis transformation as given by (3.5):

(i) $$\kappa_4 = Span[S] = \left\{ |S_{HH}|^2 + |S_{HV}|^2 + |S_{VH}|^2 + |S_{VV}|^2 \right\} = \left\{ |S_{ii}|^2 + |S_{ij}|^2 + |S_{ji}|^2 + |S_{jj}|^2 \right\} \tag{3.12}$$

confirms that the total power is conserved, and it is known as Kennaugh's span-invariant κ_4;

(ii) $S_{HV} - S_{VH} = S_{ij} - S_{ji}$, for monostatic case (3.13)

warranting symmetry of the scattering matrix in any polarization basis as long as the BSA for the strictly mono-static but not general bistatic case is implied;

(iii) $Det\{[S]_{HV}\} = Det\{[S]_{ij}\}$ or $Det\{[S(HV)]\} = Det\{[S(ij)]\}$ (3.14)

due to the fact that $Det\{[U_2]\} = 1$ implies determinantal invariance.

In addition, diagonalization of the scattering matrix, also for the general bistatic case, can always be obtained but requires mixed basis representations by using the *'Singular Value Decomposition Theorem (SVD)'* [52, 53] so that the diagonalized scattering matrix $[S_D]$ can be obtained by the left and right singular vectors, where

$$[S_D] = [Q_L][S][Q_R] \text{ with } [S_D] = \begin{bmatrix} S_{\lambda_1} & 0 \\ 0 & S_{\lambda_2} \end{bmatrix} \tag{3.15}$$

and $|Det\{[Q_L]\}| = |Det\{[Q_R]\}| = 1$

and S_{λ_1} and S_{λ_2} denote the diagonal eigenvalues of $[S]$, and the diagonal elements S_{λ_1} and S_{λ_2} can be taken as real nonnegative and are known as the singular values of the matrix $[S]$. For the symmetric scattering matrices in the mono-static case (MSA), diagonalization is achieved according to the *unitary consimilarity* transform for which

$$[Q_R] = [Q_L]^T \tag{3.16}$$

and above equations will simplify due to the restriction of symmetry $S_{ij} = S_{ji}$.

3.4 The 4x4 Mueller (Forward Scattering) [*M*] and the 4x4 Kennaugh (Back-Scattering) [*K*] Power Density Matrices

For the partially polarized cases, there exists an alternate formulation of expressing the scattered wave in terms of the incident wave via the 4x4 Mueller$[M]$ and Kennaugh $[K]$ matrices for the FSA and BSA coordinate formulations, respectively, where

$$[q^s] = [M][q^i] \tag{3.17}$$

For the purely coherent case, $[M]$ can formally be related to the coherent Jones Scattering Matrix $[T]$ as

$$[M] = \begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix}[A]^{T-1}([T]\otimes[T]^*)[A]^{-1} = [A]([T]\otimes[T]^*)[A]^{-1} \tag{3.18}$$

with $\otimes$ symbolizing the standard Kronecker tensorial matrix product relations [112] provided in (A.1), Appendix A, and the 4x4 expansion matrix $[A]$ is given by [76] as

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix} \tag{3.19}$$

with the elements M_{ij} of $[M]$, given in (B.1), Appendix B.

Specifically we find that if $[T]$ is normal, i.e. $[T][T]^{T*} = [T]^{T*}[T]$, then $[M]$ is also normal, i.e. $[M][M]^T = [M]^T[M]$.

Similarly, for the purely coherent case [76], $[K]$ can formally be related to the *coherent Sinclair matrix* $[S]$ as

$$[K] = 2[A]^{T-1}([S] \otimes [S]^*)[A]^{-1} \quad (3.20)$$

where

$$[A]^{T-1} = \frac{1}{2}[A]^* \quad (3.21)$$

and for a symmetric Sinclair matrix $[S]$, then $[K]$ is symmetric, keeping in mind the '*mathematical formalism'* $[M] = diag[1 \quad 1 \quad 1 \quad -1][K]$, but great care must be taken in strictly distinguishing the physical meaning of $[K]$ versus $[M]$ in terms of $[S]$ versus $[T]$ respectively. Thus, if $[S]$ is symmetric, $S_{HV} = S_{VH}$, then $[K]$ is symmetric, $K_{ij} = K_{ji}$; and the correct elements for $[M]$, $[K]$ and the symmetric cases are presented in (B.1 – B.5), Appendix B.

3.5 The 2x2 Graves Polarization Power Scattering Matrix [*G*]

Kennaugh introduces, next to the Kennaugh matrix $[K]$, another formulation $\left[G\right]$, for expressing the power in the scattered wave $\mathbf{E}^S$ to the incident wave $\mathbf{E}^i$ for the coherent case in terms of the so-called '*Graves polarization coherent power scattering matrix* $[G]$ *'*, where

$$P^S = \frac{1}{8\pi\, Z_0 r_2^2} \mathrm{E}^{iT^*}[G]\mathrm{E}^i \quad (3.22)$$

so that in terms of the Kennaugh elements K_{ij}, defined in the appendix, for the mono-static case

$$[G] = \langle [S]^{T^*}[S] \rangle = \begin{bmatrix} K_{11} + K_{12} & K_{13} - jK_{14} \\ K_{13} + jK_{14} & K_{11} - K_{12} \end{bmatrix} \quad (3.23)$$

By using a single coordinate system for (x_1, y_1, z_1) and (x_3, y_3, z_3) for the monostatic case, as in Fig. 3.1, and also described in detail in [19], it can be shown that for a scatterer ensemble (e.g. precipitation) for which individual scatterers move slowly compared to a period of the illuminating wave, and quickly compared to the time-averaging of the receiver, time-averaging can be adjusted to find the decomposed power scattering matrix $\langle [G] \rangle$, as

$$\langle [G] \rangle = \langle [S(t)]^{T^*}[S(t)] \rangle = [G_H] + [G_V] = \left\{ \left\langle \begin{bmatrix} |S_{HH}|^2 & S_{HH}^* S_{HV} \\ S_{HH} S_{HV}^* & |S_{HV}|^2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} |S_{VH}|^2 & S_{VH}^* S_{VV} \\ S_{VH} S_{VV}^* & |S_{VV}|^2 \end{bmatrix} \right\rangle \right\} \quad (3.24)$$

This shows that the time averaged *'Graves Power Scattering Matrix* $\langle [G] \rangle$*'*, first introduced by Kennaugh [4, 5], can be used to divide the powers that are received by linear horizontally and vertically by polarized antennas, as discussed in more detail in [19] and in [113]. It should be noted that a similar decomposition also exists for the Muller/Jones matrices, commonly denoted as FSA power scattering matrix

$$\langle [F] \rangle = \left\langle [T(t)]^{+} [T(t)] \right\rangle = [F_H] + [F_V] = \left\{ \left\langle \begin{bmatrix} |T_{HH}|^2 & T_{HH}^* T_{HV} \\ T_{HH} T_{HV}^* & |T_{HV}|^2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} |T_{VH}|^2 & T_{VH}^* T_{VV} \\ T_{VH} T_{VV}^* & |T_{VV}|^2 \end{bmatrix} \right\rangle \right\} \quad (3.25)$$

which is not further analyzed here [113].

3.6 Co/Cross-Polar Backscattering Power Decomposition for the One-Antenna (Transceiver) and the Matched Two-Antenna (Quasi-Monostatic) Cases

Assuming that the scatterer is placed in free unbounded space and that no polarization state transformation occurs along the propagation path from the transmitter (T) to the scatterer incidence (S), and along that from the scatterer(s) to the receiver (R), then the value of the terminal voltage of the receiver, V_R, induced by an arbitrarily scattered wave $\mathbf{E}_R$ at the receiver, is defined by the r*adar brightness function* V_R, and the corresponding received power P_R expression

$$V_R = \hat{\mathbf{h}}_{\mathbf{R}}^T \, \mathbf{E}_{\mathbf{R}} \qquad P_R = \frac{1}{2} V_R^{\,*} \, V_R \quad (3.26)$$

with the definition of the Kennaugh matrix $[K]$ in terms of the Sinclair matrix $[S]$, the received power or *radar brightness* function may be re-expressed

$$P_{RT} = \frac{1}{2} \left| \hat{\mathbf{h}}_R \, [\mathrm{S}] \, \mathbf{E}_T \right|^2 = \frac{1}{2} \, \mathbf{q}_R^T \, [K] \, \mathbf{q}_T \quad (3.27)$$

where $\mathbf{q}_R$ and $\mathbf{q}_T$ the corresponding normalized Stokes' vectors.

For the one-antenna (transceiver) case the co-polar channel (c) and the cross-polar channel (x) powers become:

$$P_c = \frac{1}{2} \left| \hat{\mathbf{h}}_T^{\,T} \, [\mathbf{S}] \, \mathbf{E}_T \right|^2 = \frac{1}{2} \mathbf{q}_T^{\,T} \, [K_c] \, \mathbf{q}_T \quad (3.28)$$

$$P_x = \frac{1}{2} \left| \hat{\mathbf{h}}_{T_\perp}^{\,T} \, [\mathrm{S}] \, \mathrm{E}_T \right|^2 = \frac{1}{2} \, \mathbf{q}_T^{\,T} \, [K_x] \, \mathbf{q}_T \quad (3.29)$$

with

$$[K_c] = \left([A]^{-1}\right)^T ([T] \otimes [T]^*) [A]^{-1} = [C][K] \quad (3.30)$$

$$[K_x] = \left([A]^{-1}\right)^T ([Y][T] \otimes [T]^*) [A]^{-1} = [C_x][K] \quad (3.31)$$

and

$$[C]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\quad [Y]=\begin{bmatrix}0&0&0&1\\0&-1&0&0\\0&0&-1&0\\1&0&0&0\end{bmatrix}\quad [X]=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix} \tag{3.32}$$

For the *Two-Antenna Dual Polarization* case, in which one antenna serves as a transmitter and the other as the receiver, the optimal received power P_m for the *'matched case'* becomes by using the matching condition

$$\hat{\mathbf{h}}_{\mathbf{R}_{\mathbf{m}}} = \mathbf{E}_{\mathbf{s}}^{*} / \|\mathbf{E}_{\mathbf{s}}\| \tag{3.33}$$

so that

$$P_m = \mathbf{q}_T^{\;T}\,[K_m]\,\mathbf{q}_T\,,\ \text{where}\ [K_m]=[K_c]+[K_x]=[K_{11}][K],\ \text{and}\ [K_{11}]=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{3.34}$$

which represent an essential relationship for determining the optimal polarization states from the optimization of the Kennaugh matrix.

3.7 The Scattering Feature Vectors : The Lexicographic and the Pauli Feature Vectors

Up to now we have introduced three descriptions of the scattering processes in terms of the 2x2 Jones versus Sinclair, $[T]$ versus $[S]$, the 2x2 power scattering matrices, $[F]$ versus $[G]$, and the 4x4 power density Muller versus Kennaugh matrices, $[M]$ versus $[K]$. Alternatively, the polarimetric scattering problem can be addressed in terms of a vectorial feature descriptive formulation [114] borrowed from vector signal estimation theory. This approach replaces the 2x2 scattering matrices $[T]$ versus $[S]$, the 2x2 power scattering matrices $[F]$ versus $[G]$, and the 4x4 Muller $[M]$ versus Kennaugh $[K]$ matrices by an equivalent four-dimensional complex scattering feature vector $\mathbf{f}_4$, formally defined for the general bi-static case as

$$[S]_{HV} = \begin{bmatrix}S_{HH} & S_{HV}\\ S_{VH} & S_{VV}\end{bmatrix} \Rightarrow \mathbf{f}_4 = F\{[S]\} = \frac{1}{2}Trace\{[S]\ \psi\} = \begin{bmatrix}f_0 & f_1 & f_2 & f_3\end{bmatrix}^T \tag{3.35}$$

where $F\{[S]\}$ is the matrix vectorization operator $Trace\{[S]\}$ is the sum of the diagonal elements of $[S]$, and ψ is a complete set of 2x2 complex basis matrices under a hermitian inner product. For the vectorization of any complete orthonormal basis set [97] of four 2x2 matrices that leave the (Euclidean) norm of the scattering feature vector invariant, can be used, and there are two such bases favored in the polarimetric radar literature; one being the *'lexicographic basis'* $[\Psi_L]$, and the other *'Pauli spin matrix set'* $[\Psi_P]$. We note here that the distinction between the lexicographic and Pauli-based feature vector representation is related to Principal and Independent Component Analysis (PCA/ICA) which is an interesting topic for future research.

(i) The *'Lexicographic Feature vector'*: $\mathbf{f}_{4L}$, is obtained from the simple lexicographic expansion of $[S]$ using $[\Psi_L]$, with

$$[\Psi_L] \equiv \left\{ 2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \; 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \; 2\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \; 2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\} \tag{3.36}$$

so that the corresponding feature vector becomes

$$\mathbf{f}_{4L} = [S_{HH} \quad S_{HV} \quad S_{VH} \quad S_{VV}]^T \tag{3.37}$$

(ii) The Pauli Feature vector $\mathbf{f}_{4P}$ is obtained from the renowned complex Pauli spin matrix basis set $[\Psi_P]$ which in a properly re-normalized presentation is here defined as

$$[\Psi_P] \equiv \left\{ \sqrt{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \; \sqrt{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \; \sqrt{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \; \sqrt{2}\begin{bmatrix}0 & -j\\j & 0\end{bmatrix} \right\} \tag{3.38}$$

resulting in the *'polarimetric correlation phase'* preserving *'Pauli Feature Vector'*.

$$\mathbf{f}_{4P} = [f_0 \quad f_1 \quad f_2 \quad f_3]_P^T = \frac{1}{\sqrt{2}}[S_{HH}+S_{VV} \quad S_{VV}-S_{HH} \quad S_{HV}+S_{VH} \quad j(S_{HV} - S_{VH})]^T \tag{3.39}$$

where the corresponding scattering matrix $[S]_P$ is related to the $\mathbf{f}_{4P} = [f_0 \quad f_1 \quad f_2 \quad f_3]_P^T$ by

$$[S]_P = \frac{1}{\sqrt{2}}\begin{bmatrix} f_0 - f_1 & f_2 - jf_3 \\ f_2 + jf_1 & f_0 + f_1 \end{bmatrix} = [S] \tag{3.40}$$

3.8 The Unitary Transformations of the Feature Vectors

The insertion of the factor 2 in (3.36) versus the factor $\sqrt{2}$ in (3.38) arises from the *'total power invariance'*, i.e. keeping the norm independent from the choice of the basis matrices Ψ, so that

$$\|\mathbf{f}_4\| = \mathbf{f}_4^\dagger \cdot \mathbf{f}_4 = \frac{1}{2}\,\mathrm{Span}\{[S]\} = \frac{1}{2}\,Trace\{[S][S]^\dagger\} = \frac{1}{2}\left(|S_{HH}|^2 + |S_{HV}|^2 + |S_{VH}|^2 + |S_{VV}|^2\right) = \kappa_4 \tag{3.41}$$

This constraint forces the transformation from the lexicographic to the Pauli-based feature vector [52, 53, 114], or to any other desirable one, to be unitary, where with

$$\mathbf{f}_{4P} = [D_4]\mathbf{f}_{4L} \qquad \text{and reversely} \qquad \mathbf{f}_{4L} = [D_4]^{-1}\mathbf{f}_{4P} \tag{3.42}$$

we find

$$[D_4] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 1\\1 & 0 & 0 & -1\\0 & 1 & 1 & 0\\0 & j & -j & 0\end{bmatrix} \qquad [D_4]^{-1} = [D_4]^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 1 & -j\\0 & 0 & 1 & j\\1 & -1 & 0 & 0\end{bmatrix} \tag{3.43}$$

Furthermore, these special unitary matrices relating the feature vectors control the more general cases of transformations related to the change of polarization basis. By employing the Kronecker direct tensorial product of matrices, symbolized by $\otimes$, we obtain, the transformation for the scattering vector from the linear $\{\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_V\}$ to any other elliptical polarization basis $\{\hat{\mathbf{u}}_A, \hat{\mathbf{u}}_B\}$, characterized by the complex polarization ratio by

$$\mathbf{f}_{4L}(AB) = [U_{4L}]\mathbf{f}_{4L}(HV) \quad \text{and} \quad \mathbf{f}_{4P}(AB) = [U_{4P}]\mathbf{f}_{4P}(HV) \tag{3.44}$$

where $[U_{4L}]$ is the transformation matrix for the conventional feature vector $\mathbf{f}_{4L}$

Here we note that in order to obtain the expression $[U_{4L}] = [U_2] \otimes [U_2]^T$, the unitary congruence (unitary consimilarity) transformation for the Sinclair scattering matrix in the reciprocal case was used. This implies however that we must distinguish between forward scattering and backscattering (and so also bistatic scattering); where for the reciprocal backscatter case the 3-dimensional target feature vectors ought to be used. These features lead to interesting questions which however need more in depth analyses for which the ubiquity of the Time Reversal operation shows up again.

$$[U_{4L}] = [U_2] \otimes [U_2]^T = \frac{1}{1+\rho\rho^*} \begin{bmatrix} 1 & -\rho^* & -\rho^* & \rho^{*^2} \\ \rho & 1 & -\rho\rho^* & -\rho^* \\ \rho & -\rho\rho^* & 1 & -\rho^* \\ \rho^2 & \rho & \rho & 1 \end{bmatrix} \tag{3.45}$$

and $[U_{4P}]$ is the homologous transformation matrix for the Pauli-based feature vector $\mathbf{f}_{4P}$

$$[U_{4P}] = [D_4][U_{4L}][D_4]^\dagger \tag{3.46}$$

where $[U_{4L}]$ and $[U_{4P}]$ are special 4x4 unitary matrices for which with $[I_4]$ denoting the 4x4 identity matrix

$$[U_4][U_4] = [I_4] \quad \text{and} \quad Det\{[U_4]\} = 1 \tag{3.47}$$

Kennaugh matrices and covariance matrices are based on completely different concepts (notwithstanding their formal relationships) and must be clearly separated which is another topic for future research.

The main advantage of using the scattering feature vector, $\mathbf{f}_{4L}$ or $\mathbf{f}_{4P}$, instead of the Sinclair scattering matrix $[S]$ and the Kennaugh matrix $[K]$, is that it enables the introduction of the covariance matrix decomposition for partial scatterers of a dynamic scattering environment. However, there does not exist a physical but only a strict relationship mathematical between the two alternate concepts for treating the partially coherent case, which is established and needs always to be kept in focus [114]. It should be noted that besides the covariance matrices the so-called (normalized) correlation matrices are often used advantageously especially when the eigenvalues of a covariance matrix have large variations.

3.9 The Polarimetric Covariance Matrix

In most radar applications, the scatterers are situated in a dynamically changing environment and are subject to spatial (different view angles as in 'SAR') and temporal variations (different hydro-meteoric states in 'RAD-MET'), if when illuminated by a monochromatic waves cause the back-scattered wave to be partially polarized with incoherent scattering contributions so that "$\langle [S] \rangle = \langle [S(\mathbf{r},t)] \rangle$". Such scatterers, analogous to the partially polarized waves are called partial scatterers [78, 90]. Whereas the Stokes vector, the wave coherency matrix, and the Kennaugh/Mueller matrix representations provide a first approach of dealing with partial scattering descriptions, the unitary matrix derived from the scattering feature $\mathbf{f}_4$ vector provides another approach borrowed from decision and estimation signal theory [115] which are currently introduced in Polarimetric SAR and Polarimetric-Interferometric SAR analyses, and these need to be introduced here. However, even if the environment is dynamically changing one has to make assumption concerning stationarity (at least over timescales of interest), homogeneity and ergodicity. This can be analyzed more precise by introducing the concept of space and time varying stochastic processes.

The 4x4 lexicographic polarimetric covariance matrix $[C_{4L}]$ and the Pauli-based covariance matrix $[C_{4P}]$ are defined, using the outer product $\otimes$ of the feature vector with its conjugate transpose as

$$[C_{4L}] = \langle \mathbf{f}_{4L} \cdot \mathbf{f}_{4L}^{\dagger} \rangle \quad \text{and} \quad [C_{4P}] = \langle \mathbf{f}_{4P} \cdot \mathbf{f}_{4P}^{\dagger} \rangle \tag{3.48}$$

where $\langle ... \rangle$ indicates temporal or spatial ensemble averaging, assuming homogeneity of the random medium. The lexicographic covariance matrix $[C_4]$ contains the complete information in amplitude and phase variance and correlation for all complex elements of $[S]$ with

$$[C_{4L}] = \langle \mathbf{f}_{4L} \cdot \mathbf{f}_{4L}^{\dagger} \rangle = \begin{bmatrix} \langle |S_{HH}|^2 \rangle & \langle S_{HH} S_{HV}^* \rangle & \langle S_{HH} S_{VH}^* \rangle & \langle S_{HH} S_{VV}^* \rangle \\ \langle S_{HV} S_{HH}^* \rangle & \langle |S_{HV}|^2 \rangle & \langle S_{HV} S_{VH}^* \rangle & \langle S_{HV} S_{VV}^* \rangle \\ \langle S_{VH} S_{HH}^* \rangle & \langle S_{VH} S_{HV}^* \rangle & \langle |S_{VH}|^2 \rangle & \langle S_{VH} S_{VV}^* \rangle \\ \langle S_{VV} S_{HH}^* \rangle & \langle S_{VV} S_{HV}^* \rangle & \langle S_{VV} S_{VH}^* \rangle & \langle |S_{VV}|^2 \rangle \end{bmatrix} \tag{3.49}$$

and

$$[C_{4P}] = \langle \mathbf{f}_{4P} \cdot \mathbf{f}_{4P}^{\dagger} \rangle = \langle [D_4]\mathbf{f}_{4L} \cdot \mathbf{f}_{4L}^{\dagger} [D_4]^{\dagger} \rangle = [D_4] \langle \mathbf{f}_{4L} \cdot \mathbf{f}_{4L}^{\dagger} \rangle [D_4]^{\dagger} = [D_4][C_{4L}][D_4]^{\dagger} \tag{3.50}$$

Both the *'Lexicographic Covariance* $[C_{4L}]$ ' and the '*Pauli-based Covariance* $[C_{4P}]$*]'* matrices are hermitian positive semi-definite matrices which implies that these possess real non-negative eigenvalues and orthogonal eigenvectors. Incidentally, those can be mathematically related directly to the Kennaugh matrix $[K]$, which is not shown here; however, there does not exist a physical relationship between the two presentations which must always be kept in focus.

The transition of the covariance matrix from the particular linear polarization reference basis {H V} into another elliptical basis {A B}, using the change-of-basis transformations defined in (3.41 – 3.45), where for

$$[C_{4L}(AB)] = \langle \mathbf{f}_{4L}(AB) \cdot \mathbf{f}_{4L}^{\dagger}(AB) \rangle = [U_4] \langle \mathbf{f}_{4L}(HV) \cdot \mathbf{f}_{4L}^{\dagger}(HV) \rangle [U_4]^{\dagger} = [D_4][C_{4L}(HV)][D_4]^{\dagger} \tag{3.51}$$

and for

$$[C_{4P}(AB)] = \langle \mathbf{f}_{4P}(AB) \cdot \mathbf{f}_{4P}^{\dagger}(AB) \rangle = [U_4] \langle \mathbf{f}_{4P}(HV) \cdot \mathbf{f}_{4P}^{\dagger}(HV) \rangle [U_4]^{\dagger} = [D_4][C_{4P}(HV)][D_4]^{\dagger} \tag{3.52}$$

The lexicographic and Pauli-based covariance matrices, $[C_{4L}]$ and $[C_{4P}]$, contain, in the most general case, as defined in (3.49) and (3.50), sixteen independent parameters, namely four real power densities and six complex phase correlation parameters.

3.10 The Monostatic Reciprocal Back-Scattering Cases

For a reciprocal target matrix, in the mono-static (backscattering) case, the reciprocity constrains the Jones matrix to be usually normal, and the Sinclair scattering matrix to be symmetrical, i.e. $S_{HV} = S_{VH}$, which further reduces the expressions of $[G]$ and $[K]$. Furthermore, the four-dimensional scattering feature vector $\mathbf{f}_4$ reduces to a three- dimensional scattering feature vector $\mathbf{f}_3$ such that following [97]

$$\mathbf{f}_{3L} = [Q] \; , \; \mathbf{f}_{4L} = \begin{bmatrix} S_{HH} & \sqrt{2}S_{HV} & S_{VV} \end{bmatrix}^T \; , \; S_{HV} = S_{VH} \tag{3.53}$$

where with $[I_3]$ being the unit 3x3 matrix and always keeping in mind that the transformation between lexicographic and Pauli ordering is a direct transformation of the scattering matrix (and not only of the covariance matrices)

$$[Q] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [Q][Q]^T = [I_3] \tag{3.54}$$

and the factor $\sqrt{2}$ needs to be retained in order to keep the vector norm consistent with the span invariance κ.

Similarly, the reduced Pauli feature vector $\mathbf{f}_{3P}$ becomes

$$\mathbf{f}_{3P} = [Q] \; , \mathbf{f}_{4P} = \frac{1}{\sqrt{2}} \begin{bmatrix} S_{HH} + S_{VV} & S_{HH} - S_{VV} & 2S_{HV} \end{bmatrix}^T \; , \; S_{HV} = S_{VH} \tag{3.55}$$

The three-dimensional scattering feature vector from the lexicographic to the Pauli-based matrix basis, and vice versa, are related as

$$\mathbf{f}_{3P} = [D_3]\mathbf{f}_{3L} \quad \text{and} \quad \mathbf{f}_{3L} = [D_3]^{-1}\mathbf{f}_{3P} \tag{3.56}$$

with $[D_3]$ defining a special 3x3 unitary matrix

$$[D_3] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \quad \text{and} \quad [D_3]^{-1} = [D_3]^{\dagger} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{bmatrix} \tag{3.57}$$

The change-of-basis transformation for the reduced scattering vectors in terms of the complex polarization ratio ρ of the new basis is given by

$$\mathbf{f}_{3L}(AB) = [U_{3L}(\rho)]\mathbf{f}_{3L}(HV) \quad \text{and} \quad \mathbf{f}_{3P}(AB) = [U_{3P}(\rho)]\mathbf{f}_{3P}(HV) \tag{3.58}$$

where

$$[U_{3L}] = \frac{1}{1+\rho\rho^*} \begin{bmatrix} 1 & \sqrt{2}\rho & \rho^2 \\ -\sqrt{2}\rho^* & 1-\rho\rho^* & \sqrt{2}\rho \\ \rho^{*2} & -\sqrt{2}\rho^* & 1 \end{bmatrix} \tag{3.59}$$

and

$$[U_{3P}] = [D_3][U_{3L}][D_3]^{\dagger} = \frac{1}{2(1+\rho\rho^*)}\begin{bmatrix} 2+\rho^2+\rho^{*^2} & \rho^{*^2}-\rho^2 & 2(\rho-\rho^*) \\ \rho^2-\rho^{*^2} & 2-(\rho^2+\rho^{*^2}) & 2(\rho+\rho^*) \\ 2(\rho-\rho^*) & -2(\rho+\rho^*) & 2(1-\rho\rho^*) \end{bmatrix} \tag{3.60}$$

which are 3x3 special unitary matrices.

Thus, a reciprocal scatterer is completely described either by the 3x3 *'Polarimetric Covariance Matrix* $[C_{3L}]$'

$$[C_{3L}] = \langle \mathbf{f}_{3L} \cdot \mathbf{f}_{3L}^{\dagger} \rangle = \begin{bmatrix} \langle |S_{HH}|^2 \rangle & \sqrt{2}\langle S_{HH}S_{HV}^* \rangle & \langle S_{HH}S_{VV}^* \rangle \\ \sqrt{2}\langle S_{HV}S_{HH}^* \rangle & 2\langle |S_{HV}|^2 \rangle & \sqrt{2}\langle S_{HV}S_{VV}^* \rangle \\ \langle S_{VV}S_{HH}^* \rangle & \sqrt{2}\langle S_{VV}S_{HV}^* \rangle & \langle |S_{VV}|^2 \rangle \end{bmatrix} \tag{3.61}$$

or by the 3x3 '*Polarimetric Pauli Coherency Matrix* $[C_{3P}]$'

$$[C_{3P}] = \langle \mathbf{f}_{3P} \cdot \mathbf{f}_{3P}^{\dagger} \rangle = \frac{1}{2}\begin{bmatrix} \langle |S_{HH}+S_{VV}|^2 \rangle & \langle (S_{HH}+S_{VV})(S_{HH}-S_{VV})^* \rangle & 2\langle (S_{HH}+S_{VV})S_{HV}^* \rangle \\ \langle (S_{HH}-S_{VV})(S_{HH}+S_{VV})^* \rangle & \langle |S_{HH}-S_{VV}|^2 \rangle & 2\langle (S_{HH}-S_{VV})S_{HV}^* \rangle \\ 2\langle S_{HV}(S_{HH}+S_{VV})^* \rangle & 2\langle S_{HV}(S_{HH}-S_{VV})^* \rangle & 4\langle |S_{HV}|^2 \rangle \end{bmatrix} \tag{3.62}$$

where the relation between the 3x3 Pauli coherency matrix $[C_{3P}]$ and the 3x3 covariance matrix $[C_{3L}]$ is given by

$$[C_{3L}] = \langle \mathbf{f}_{3L} \cdot \mathbf{f}_{3L}^{\dagger} \rangle = \langle [D_3]\mathbf{f}_{3P} \cdot \mathbf{f}_{3P}^{\dagger}[D_3]^{\dagger} \rangle = [D_3]\langle \mathbf{f}_{3P} \cdot \mathbf{f}_{3P}^{\dagger} \rangle [D_3]^{\dagger} = [D_3][C_{3P}][D_3]^{\dagger} \tag{3.63}$$

and

$$[C_{3L}(AB)] = \langle \mathbf{f}_{3L}(AB) \cdot \mathbf{f}_{3L}^{\dagger}(AB) \rangle = [U_{3L}]\langle \mathbf{f}_{3L}(HV) \cdot \mathbf{f}_{3L}^{\dagger}(HV) \rangle [U_{3L}]^{\dagger} = [U_{3L}][C_{3L}(HV)][U_{3L}]^{\dagger} \tag{3.64}$$

$$[C_{3P}(AB)] = \langle \mathbf{f}_{3P}(AB) \cdot \mathbf{f}_{3P}^{\dagger}(AB) \rangle = [U_{3P}]\langle \mathbf{f}_{3P}(HV) \cdot \mathbf{f}_{3P}^{\dagger}(HV) \rangle [U_{3P}]^{\dagger} = [U_{3P}][C_{3P}(HV)][U_{3P}]^{\dagger} \tag{3.65}$$

where

$$[U_{3L}(\rho)][U_{3L}(\rho)]^{+} = [I_3] \quad \text{and} \quad Det\{[U_{3L}(\rho)]\} = 1 \tag{3.66}$$

and

$$\|\mathrm{f}_{3L}\|^2 = \|\mathrm{f}_{3P}\|^2 = \frac{1}{2}\,Span\,\{[S]\} = \frac{1}{2}\,Trace\{[S][S]^{\dagger}\} = \frac{1}{2}\,\left\{|S_{HH}|^2 + 2|S_{HV}|^2 + |S_{VV}|^2\right\} = \kappa_3 \tag{3.67}$$

3.11 Co/Cross-polar Power Density and Phase Correlation Representations

The Covariance matrix elements are directly related to polarimetric radar measurables, comprised of the *Co/Cross-Polar Power Densities* $P_c(\rho)$, $P_x(\rho)$, $P_c^{\perp}(\rho)$, and the *Co/Cross-Polar Phase Correlations* $R_c(\rho)$, $R_x(\rho)$, $R_x^{\perp}(\rho)$, [81], *where*

$$[C_{3L}(\rho)] = \begin{bmatrix} P_c(\rho) & \sqrt{2}\,R_x(\rho) & R_c(\rho) \\ \sqrt{2}\,R_x(\rho)^* & 2P_x(\rho) & \sqrt{2}\,R_x^{\perp}(\rho)^* \\ R_c(\rho)^* & \sqrt{2}\,R_x^{\perp}(\rho) & P_c^{\perp}(\rho) \end{bmatrix} \tag{3.68}$$

Once the covariance matrix has been measured in one basis, e.g., $[C_{3L}(H,V)]$ in {H V} basis, it can easily be determined analytically for any other basis by definition of (3.60). Plotting the mean power returns and phase correlations as function of the complex polarization ratio ρ or the geometrical polarization ellipse parameters ψ, χ, of (3), yields the familiar '*polarimetric signature plots'*. In addition, the expressions for the degree of coherence $\mu(\rho)$ and polarization $D_p(\rho)$ defined in (2.30) and (2.31), respectively are given according to [34] by

$$\mu(\rho) = \frac{|R_x(\rho)|}{\sqrt{P_c(\rho)P_x(\rho)}}, \quad D_p(\rho) = \frac{\left\{[P_c(\rho) - P_x(\rho)]^2 + 4|R_x(\rho)|^2\right\}^{1/2}}{(P_c(\rho) + P_x(\rho))}, \quad where \;\; 0 \le \mu(\rho) \le D_p(\rho) \le 1 \tag{3.69}$$

and for coherent (deterministic) scatterers $\mu = D_p = 1$, whereas for completely depolarized scatterers $\mu = D_p = 0$.

The covariance matrix possesses additional valuable properties for the reciprocal back-scattering case which can be demonstrated by transforming $[C_{3L}(H,V)]$ into its orthogonal representation for $\rho_{\perp} = \left({-1}/{\rho^*} \right)$ so that

$$\left[C_{3L}\left(\rho_{\perp} = {-1}/{\rho^*}\right)\right] = \begin{bmatrix} P_c^{\perp}(\rho) & \frac{-\rho}{\rho^*}\sqrt{2}\,R_x^{\perp}(\rho) & \frac{\rho^2}{\rho^{*2}} R_c(\rho) \\ \frac{-\rho^*}{\rho}\sqrt{2}\,R_x^{\perp}(\rho)^* & 2P_x(\rho) & \frac{-\rho}{\rho^*}\sqrt{2}\,R_x(\rho)^* \\ \frac{\rho^{*2}}{\rho^2} R_c(\rho)^* & \frac{-\rho^*}{\rho}\sqrt{2}\,R_x(\rho) & P_c(\rho) \end{bmatrix} \tag{3.70}$$

leading to the following inter-channel relations

$$P_c\left(\rho_{\perp} = {-1}/{\rho^*}\right) = P_c^{\perp}(\rho) \qquad \left|R_x\left(\rho_{\perp} = {-1}/{\rho^*}\right)\right| = \left|R_x^{\perp}(\rho)\right| \tag{3.71}$$

and the symmetry relations

$$P_x\left(\rho_{\perp} = {-1}/{\rho^*}\right) = P_x(\rho) \qquad \left|R_c\left(\rho_{\perp} = {-1}/{\rho^*}\right)\right| = \left|R_c(\rho)\right| \tag{3.72}$$

Similar, but not identical relations, could be established for the Pauli-Coherency Matrix $[C_{3P}(\rho)]$, which are not presented here. There exists another polarimetric covariance matrix representation in terms of the so-called polarimetric inter-correlation parameters σ_0, ρ, δ, β, γ, and ε, where according to [19, Chapter 5]

$$[C_{3L}] = \begin{bmatrix} 1 & \beta\sqrt{\delta} & \rho\sqrt{\gamma} \\ \beta^*\sqrt{\delta} & \delta & \varepsilon\sqrt{\gamma\,\delta} \\ \rho^*\sqrt{\gamma} & \varepsilon^*\sqrt{\gamma\,\delta} & \gamma \end{bmatrix} \tag{3.73}$$

with the polarimetric inter-correlation parameters σ_0, ρ, δ, β, γ, and ε defined as

$$\begin{aligned} \sigma_0 &= \left\langle |S_{HH}|^2 \right\rangle & \rho &= \frac{\left\langle S_{HH} S_{VV}^* \right\rangle}{\sigma_0\sqrt{\gamma}} = \frac{\left\langle S_{HH} S_{VV}^* \right\rangle}{\sqrt{\left\langle |S_{HH}|^2 \right\rangle \left\langle |S_{VV}|^2 \right\rangle}} \\ \delta &= 2\frac{\left\langle |S_{HV}|^2 \right\rangle}{\sigma_0} & \beta &= \frac{\sqrt{2}\left\langle S_{HH} S_{HV}^* \right\rangle}{\sigma_0\sqrt{\delta}} = \frac{\left\langle S_{HH} S_{HV}^* \right\rangle}{\sqrt{\left\langle |S_{HH}|^2 \right\rangle \left\langle |S_{HV}|^2 \right\rangle}} \\ \gamma &= 2\frac{\left\langle |S_{VV}|^2 \right\rangle}{\sigma_0} & \varepsilon &= \frac{\sqrt{2}\left\langle S_{HV} S_{VV}^* \right\rangle}{\sigma_0\sqrt{\delta\gamma}} = \frac{\left\langle S_{HV} S_{VV}^* \right\rangle}{\sqrt{\left\langle |S_{HV}|^2 \right\rangle \left\langle |S_{VV}|^2 \right\rangle}} \end{aligned} \tag{3.74}$$

This completes the introduction of the pertinent polarimetric matrix presentations, commonly used in radar polarimetry and in polarimetric SAR interferometry, where in addition the polarimetric interference matrices need to be introduced as shown in [19], after introducing briefly basic concepts of radar interferometry in [70].

3.12 Alternate Matrix Representations

In congruence with the alternate formulations of the of the polarization properties of electromagnetic waves, there also exist the associated alternate tensorial (matrix) formulations related to the '*four-vector Hamiltonian'* and '*spinorial'* representations as pursued by Zhivotovsky [109], and more recently by Bebbington [32]. These formulations representing most essential tools for dealing with the '*general bi-static (non-symmetric) scattering cases'* for both the coherent (*3-D Poincaré and 3-D Polarization spheroid)* and partially coherent (*4-D Zhivotovsky sphere and spheroid)* interactions, are not further pursued here; but these 'more *generalized treatments'* of radar polarimetry deserve our fullest attention.